



Carving Out the Space of Conformal Field Theories

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Abstract

We study the constraints of conformal symmetry and unitarity in Conformal Field Theories (CFTs). Crossing symmetry of four-point functions implies universal bounds on operator dimensions and three-point function coefficients. These bounds can be extracted by solving a class of infinite-dimensional convex optimization problems, giving quantitative, nonperturbative results about potentially strongly coupled theories. Our results include general bounds on operator dimensions in 4d CFTs with concrete phenomenological implications, and novel determinations of critical exponents in the 3d Ising Model. We also introduce new techniques for computing conformal blocks of higher spin operators, paving the way for further studies.

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Citations to Previously Published Work

Chapter 2 is essentially identical to

D. Simmons-Duffin, “Projectors, Shadows, and Conformal Blocks,” arXiv:1204.3894 [hep-th].

Chapter 3 is excerpted from

D. Poland and D. Simmons-Duffin, “Bounds on 4D Conformal and Superconformal Field Theories,” JHEP **1105**, 017 (2011) [arXiv:1009.2087 [hep-th]].

Chapter 4 is essentially identical to

D. Poland, D. Simmons-Duffin and A. Vichi, “Carving Out the Space of 4D CFTs,” arXiv:1109.5176 [hep-th].

Chapter 5 is essentially identical to

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¹griping about

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For Joanna.

Chapter 1

Introduction and Summary

1.1 Why Conformal Field Theories?

Conformal Field Theories (CFTs) are ubiquitous and universal. A physical system is best understood in terms of effective theories describing the relevant degrees of freedom at each energy scale $E \sim \mu$. As we change the characteristic scale μ , the parameters of the system g_i (e.g. masses and interaction strengths) evolve according to renormalization group (RG) equations,

$$\mu \frac{dg_i}{d\mu} = \beta_i(g_j). \quad (1.1)$$

In virtually all known examples, the $g_i(\mu)$ display one of two possible behaviors at low energies and long distances:

1. They blow up. This signals a breakdown in the effective description, and a boundary between two effective theories. A simple example is a massive particle, whose dimensionless mass $\mu^{-1}m$ diverges when $\mu \ll m$. We must integrate it out to obtain a new effective description involving only light degrees of freedom.
2. They approach a fixed point $g_i(\mu) \rightarrow g_{i*}$. At the fixed point, the theory is invariant under rescaling $\mu \rightarrow \lambda\mu$.

Barring the interesting possibility of an infinite cascade of effective descriptions, it's clear that the ultimate fate of a quantum theory under RG flow is scale-invariance.¹ This is already a powerful statement. But in local, Lorentz-invariant Quantum Field Theories, it has spectacular consequences. There is strong evidence that in such theories, invariance under rescaling additionally implies invariance under the larger conformal group.² The rough intuition is that a theory which is invariant under rescalings, rotations, and translations — and has local interactions — should also be invariant under transformations that everywhere “look like” a rescaling, rotation, and translation. These are conformal transformations.

Conformal Field Theories thus describe universality classes of physical systems at long distances. They also provide useful effective descriptions for theories which are not exactly scale-invariant, but whose RG evolution lingers near a fixed point over a large range of energy scales. Consequently, 4d conformal dynamics may play a crucial role in beyond the Standard Model physics. Some scenarios where it has been invoked to solve model-building problems include walking [2–7] or conformal [8–13] technicolor, explanations of the flavor hierarchies [14–17], and solutions to the supersymmetric flavor problem [18–31], the $\mu/B\mu$ problem in gauge mediation [32–37], or the η problem in inflation [38].

CFTs also describe critical phenomena in condensed matter systems, for instance liquid-vapor transitions and order-disorder transitions in magnets. The relevant CFT can be 2, 3, or 4-dimensional, depending on where the degrees of freedom are localized, and whether the description is statistical or quantum-mechanical.

Just as scale invariance emerges at low energies, it also emerges at high-energies in “asymptotically safe” theories whose effective descriptions are valid at arbitrarily short distances. Virtually

¹This scale-invariant description might be the trivial CFT if the theory is completely massive. Such theories can still have interesting non-local observables.

²See [1] for a proof of this statement in 2-dimensions. A proof in a general number of spacetime dimensions has so far been elusive.

all of the non-gravitational effective theories we study in particle physics can be realized via relevant deformations of some UV CFT. This has important mathematical as well as physical consequences. Conformal Field Theories are mathematically relatively well-controlled. For instance, as we'll see shortly, their local correlation functions can be rigorously specified by discrete combinatorial data — there is no need for path integrals or renormalization. In this sense, they can serve as building blocks for Quantum Field Theories. Thinking of general effective theories as an RG flows from CFTs has been an extremely useful paradigm, undergirding myriad advances such as the recent proof of the a -theorem [39], and discoveries of non-Lagrangian QFTs [40].

If ubiquity and universality weren't justification enough for studying Conformal Field Theories, the AdS/CFT correspondence provides an extraordinarily compelling reason. Each CFT in d -dimensions holographically encodes a UV-complete theory of gravity in AdS_{d+1} [41–43], and vice versa. In particular, Conformal Field Theory gives a complete, nonperturbative definition of string theory in AdS, and a means in principle to study trans-planckian scattering, Hawking evaporation, and other mysterious phenomena in quantum gravity. AdS/CFT also allows for holographic descriptions of phenomenological models involving branes in AdS, such as Randall-Sundrum models [44–46].

In short, Conformal Field Theories are central objects in theoretical physics. They describe universality classes in particle and condensed matter physics, provide building blocks for physical theories, and encode theories of quantum gravity. Clarifying their structure elucidates all of these diverse fields.

1.2 The Conformal Bootstrap

What is the space of CFTs? Answering the analogous question for general QFTs is impractical. But conformal symmetry is a powerful constraint. The dream of the conformal bootstrap

program is to classify CFTs using symmetries and consistency conditions alone [47–52]. The key principles have been understood for decades, and applied with success in 2d, where the infinite-dimensional Virasoro symmetry provides another powerful tool [53]. However, for CFTs in $d > 2$, we are only now learning how to obtain concrete results [54–63].

The bootstrap approach is particularly illuminating in strongly-coupled theories, where few quantitative tools exist for studying the dynamics. While it may be impossible to compute general observables, the extra symmetry present at a conformal fixed point can make calculations possible. Solving a strongly-coupled theory at long distances then becomes a question of which CFT describes its universality class. And a variety of tools are available for attacking this question — for instance, global symmetries, anomaly matching, supersymmetric indices (if applicable), and c -theorems. For example, while the 3d Ising Model has resisted an exact solution for several decades, we will see that bootstrap techniques allow for precise computations at its conformal fixed point.

The data and consistency conditions that define a CFT are simple. To compute correlators in flat space, we need only the spectrum of operator dimensions, and coefficients in the Operator Product Expansion (OPE). For example, a two-point function of a scalar \mathcal{O} with dimension Δ is determined by conformal symmetry to have the form

$$\langle \mathcal{O}(x)\mathcal{O}(0) \rangle = \frac{1}{x^{2\Delta}}. \quad (1.2)$$

Higher n -point functions are determined in terms of two-point functions by the OPE. For example, a product $\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)$ of local operators at different points can be expanded as an infinite series of local operators at a single point

$$\mathcal{O}_1(x_1)\mathcal{O}_2(x_2) = \sum_i \lambda_{12i} C_{12i}(x_{12}) \mathcal{O}_i(x_2), \quad (1.3)$$

where $x_{12} = x_1 - x_2$ is the separation between \mathcal{O}_1 and \mathcal{O}_2 . Here, the C 's are functions whose form is determined by conformal symmetry in terms of the dimensions $\Delta_{\mathcal{O}_i}$ — they are *kinematical*

quantities. Meanwhile, the coefficients λ_{12i} are *dynamical* quantities that contain information about the specific CFT.

By repeatedly applying the OPE (1.3) to pairs of operators, we can reduce an n -point function to an infinite sum of two-point functions. For example, pairing up x_1, x_2 and x_3, x_4 in a four-point function of scalars $\langle \phi(x_1) \cdots \phi(x_4) \rangle$, we obtain

$$\langle \overbrace{\phi(x_1)\phi(x_2)} \overbrace{\phi(x_3)\phi(x_4)} \rangle = \sum_{\mathcal{O} \in \phi \times \phi} \lambda_{\mathcal{O}}^2 \frac{1}{x_{12}^{2\Delta_{\phi}} x_{34}^{2\Delta_{\phi}}} g_{\mathcal{O}}(u, v), \quad (1.4)$$

where the sum is over a subset of operators (namely, *primary* operators) appearing in the $\phi \times \phi$ OPE, and $\lambda_{\mathcal{O}}$ are the associated OPE coefficients. The functions $g_{\mathcal{O}}(u, v)$ are kinematical quantities called *conformal blocks*, which depend on conformal cross ratios $u \equiv \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}$ and $v \equiv \frac{x_{23}^2 x_{14}^2}{x_{13}^2 x_{24}^2}$. They are completely determined in terms of the dimension and spin of \mathcal{O} by conformal symmetry, and in some cases explicitly known in terms of special functions.

In a sensible CFT, the OPE should be associative. Equivalently, in the example above, we should get the same result no matter which pairing of operators we use to evaluate the four-point function $\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) \rangle$,

$$\sum_{\mathcal{O}} \begin{array}{c} 1 \quad 4 \\ \diagdown \quad \diagup \\ \text{---} \mathcal{O} \text{ ---} \\ \diagup \quad \diagdown \\ 2 \quad 3 \end{array} = \sum_{\mathcal{O}} \begin{array}{c} 1 \quad 4 \\ \diagup \quad \diagdown \\ \text{---} \mathcal{O} \text{ ---} \\ \diagdown \quad \diagup \\ 2 \quad 3 \end{array},$$

or in equations,

$$\sum_{\mathcal{O} \in \phi \times \phi} \lambda_{\mathcal{O}}^2 g_{\mathcal{O}}(u, v) = \left(\frac{u}{v}\right)^{\Delta_{\phi}} \sum_{\mathcal{O} \in \phi \times \phi} \lambda_{\mathcal{O}}^2 g_{\mathcal{O}}(v, u). \quad (1.5)$$

This “crossing relation” is a nontrivial condition on the data of the CFT. Likewise, any collection of operator dimensions and OPE coefficients satisfying crossing symmetry defines a consistent CFT in flat space.

To classify CFTs, we must determine what spectrum of operators \mathcal{O} and collection of OPE coefficients $\lambda_{\mathcal{O}}$ is consistent with crossing symmetry. Unitarity places further important constraints

on this data. For example, each operator in the spectrum has a minimum dimension depending on its spin $\dim(\mathcal{O}) \gtrsim \text{spin}(\mathcal{O})$. Additionally, we can choose a basis of operators such that the OPE coefficients $\lambda_{\mathcal{O}}$ are real, and consequently the coefficients in (1.5) are positive numbers. This fact implies nontrivial bounds on the data entering the crossing relation [54], and most of this work will be concerned with understanding precisely what these bounds are.

A key insight of [54] that makes the conformal bootstrap in $d > 2$ a possibility, and not just a dream, is that these bounds can be computed by solving a certain convex optimization problem. Although exact solutions are currently out of reach, one can make significant progress with numerical methods.

1.3 Structure of this Thesis

In this thesis, we present several recent advances in applying the conformal bootstrap to CFTs in three and four dimensions. Some of the chief results are

- Completely general, nonperturbative bounds on scalar operator dimensions in three and four dimensional CFTs. In some cases, these bounds have direct phenomenological implications. For instance, we will show that models of conformal technicolor are unnatural.
- Novel determinations of critical exponents in the 3d Ising Model, with precision comparable to the best ϵ -expansion calculations and Monte-Carlo simulations.
- Novel methods for CFT computations involving higher-spin operators, opening the door for many new bootstrap studies in the future.

We begin by understanding how to compute conformal blocks $g_{\mathcal{O}}(u, v)$ which are important in the statement of crossing symmetry. In chapter 2, we introduce a method for computing conformal blocks of operators in arbitrary Lorentz representations in any spacetime dimension,

making it possible to apply bootstrap techniques to operators with spin. The key idea is to implement the “shadow formalism” of Ferrara, Gatto, Grillo, and Parisi in a setting where conformal invariance is manifest. Conformal blocks in d -dimensions can be expressed as integrals over the projective null-cone in the “embedding space” $\mathbb{R}^{d+1,1}$. Taking care with their analytic structure, these integrals can be evaluated in great generality, reducing the computation of conformal blocks to a bookkeeping exercise. To facilitate calculations in four-dimensional CFTs, we introduce techniques for writing down conformally-invariant correlators using auxiliary twistor variables, and demonstrate their use in some simple examples.

In chapter 3, we incorporate the additional constraints of $\mathcal{N} = 1$ supersymmetry in four dimensions, and compute conformal blocks for four-point functions of chiral and anti-chiral operators. These results pave the way for general bounds on operator dimensions in 4d superconformal theories.

In chapter 4, we introduce a new numerical algorithm based on semidefinite programming to efficiently compute bounds on operator dimensions, central charges, and OPE coefficients in 4d conformal and $\mathcal{N} = 1$ superconformal field theories. Using our algorithm, we dramatically improve previous bounds on a number of CFT quantities, particularly for theories with global symmetries. In the case of $\text{SO}(4)$ or $\text{SU}(2)$ symmetry, our bounds severely constrain models of conformal technicolor. In $\mathcal{N} = 1$ superconformal theories, we place strong bounds on $\dim(\Phi^\dagger\Phi)$, where Φ is a chiral operator. These bounds asymptote to the line $\dim(\Phi^\dagger\Phi) \leq 2\dim(\Phi)$ near $\dim(\Phi) \simeq 1$, forbidding positive anomalous dimensions in this region. We also place novel upper and lower bounds on OPE coefficients of protected operators in the $\Phi \times \Phi$ OPE. Finally, we find examples of lower bounds on central charges and flavor current two-point functions that scale with the size of global symmetry representations. In the case of $\mathcal{N} = 1$ theories with an $\text{SU}(N)$ flavor symmetry, our bounds on current two-point functions lie within an $O(1)$ factor of the values realized in supersymmetric QCD in the conformal window.

Finally, in chapter 5 we study the constraints of crossing symmetry and unitarity in general 3d Conformal Field Theories. In doing so we derive new results for conformal blocks appearing in four-point functions of scalars and present an efficient method for their computation in arbitrary space-time dimension. Comparing the resulting bounds on operator dimensions and OPE coefficients in 3d to known results, we find that the 3d Ising model lies at a corner point on the boundary of the allowed parameter space. We also derive general upper bounds on the dimensions of higher spin operators, relevant in the context of theories with weakly broken higher spin symmetries.

Chapter 2

Projectors, Shadows, and Conformal Blocks

This chapter is a lightly-edited version of

D. Simmons-Duffin, “Projectors, Shadows, and Conformal Blocks,” arXiv:1204.3894 [hep-th].

2.1 Introduction

Current bootstrap methods rely crucially on explicit expressions for conformal blocks, which encode the contribution of a primary operator \mathcal{O} to a four-point function of primary operators $\langle\phi_1\phi_2\phi_3\phi_4\rangle$. We will introduce a method for computing conformal blocks of operators in arbitrary Lorentz representations, making it possible to study the full implications of crossing symmetry and unitarity in CFTs. Compact expressions for scalar conformal blocks in two and four dimensions [64, 65] were important in the initial discovery of universal bounds on operator dimensions and OPE coefficients [54–56]. Later, they were essential for improving these methods and deriving bounds with concrete phenomenological implications [59–61]. Computations of scalar

superconformal blocks allowed for similar bounds in 4d superconformal theories [57, 60, 61]. More recently, an improved understanding of scalar conformal blocks in 3d led to novel determinations of operator dimensions in the 3d Ising model [63], with precision comparable to the best perturbative calculations and Monte-Carlo simulations.

All of these results come from studying crossing symmetry and unitarity for a four-point function of scalars. But this is a small subset of the full consistency conditions of a CFT. Why not study correlators of more general operators, not just scalars? For example, applying bootstrap methods to four-point functions of currents $\langle J^\mu J^\nu J^\rho J^\sigma \rangle$ or the stress-tensor $\langle T^{\mu\nu} T^{\rho\sigma} T^{\kappa\lambda} T^{\alpha\beta} \rangle$ might lead to universal bounds on symmetry representations and central charges, perhaps shedding light on the bounds on a, c in [66], or the weak gravity conjecture [67].

Such investigations would require explicit expressions for conformal blocks of operators with spin. Unfortunately, these are scarce. Methods for computing scalar blocks can become intractable in the case of higher spin. For example, Dolan and Osborn derived scalar blocks by solving an eigenvalue equation for the quadratic-Casimir of the conformal group, which takes the form of a single second-order PDE [65]. But because of the many tensor-structures that can enter a four-point function of spin-1 operators, the analogous equation for conformal blocks of spin-1 operators is a system of 43 coupled second-order PDEs.

Partial progress on this problem was made recently in [68, 69], where the authors leveraged existing results for scalar blocks to write down conformal blocks for traceless symmetric tensors (TSTs) of the Lorentz group. This is sufficient for bootstrapping 3d CFTs, where TSTs exhaust the list of bosonic Lorentz representations. However in $d > 3$, it is insufficient.

In this work, we present a general method for computing conformal blocks of operators in arbitrary Lorentz representations. The underlying idea is based on the shadow formalism of Ferrara, Gatto, Grillo, and Parisi [48–51]. Given an operator $\mathcal{O}(x)$ with dimension Δ in a d -dimensional

CFT, they define a nonlocal “shadow operator” $\tilde{\mathcal{O}}(x)$ with dimension $\tilde{\Delta} = d - \Delta$. The integral

$$\int d^d x \mathcal{O}(x) |0\rangle\langle 0| \tilde{\mathcal{O}}(x) \quad (2.1)$$

is then dimensionless and invariant under conformal transformations. When inserted between pairs of operators, it *almost* does the job of projecting onto the contribution of \mathcal{O} to a four-point function — the conformal block $g_{\mathcal{O}}$,

$$\int d^d x \langle \phi_1(x_1) \phi_2(x_2) \mathcal{O}(x) \rangle \langle \tilde{\mathcal{O}}(x) \phi_3(x_3) \phi_4(x_4) \rangle = g_{\mathcal{O}}(x_i) + \text{“shadow block”}. \quad (2.2)$$

The extra “shadow block” is distinguished from $g_{\mathcal{O}}(x_i)$ by its behavior as $x_{12} \rightarrow 0$, and needs to be subtracted off.

A challenge in applying this procedure to operators with spin is defining conformally-invariant projectors analogous to (2.1). For this, it is extremely useful to use the embedding-space formalism [47, 70–73], which makes conformal invariance manifest by linearizing the action of the conformal group. In section 2.2, we introduce the shadow formalism in this context, using scalar conformal blocks as an example but casting shadows into a form which readily generalizes to higher spin. We show how (2.1) can be understood as a manifestly conformally-invariant integral over the projective null-cone in $\mathbb{R}^{d+1,1}$, called a “conformal integral.” The utility of writing conformally-invariant integrals in projective space has already been recognized to some extent in loop calculations for amplitudes [74–76]. In this work, it will be crucial both for ensuring conformal invariance and simplifying calculations. Also in section 2.2, we give a simple way to disentangle the conformal block $g_{\mathcal{O}}(x_i)$ from its shadow by considering the action of a monodromy $x_{1,2} \rightarrow e^{2\pi i} x_{1,2}$.

In section 2.3, we compute all conformal integrals which arise in conformal block computations, and clarify their properties under monodromy. Using the embedding space, integrals with nontrivial Lorentz indices are no more difficult than scalar integrals, and the results of this section apply equally well to scalar and higher-spin blocks. In even spacetime dimensions, the expressions

are sums of products of elementary hypergeometric functions. In section 2.4, we explain the strategy for combining these results to compute higher spin conformal blocks. As an example, we write down conformally-invariant projectors for tensor operators, and compute the conformal block for the exchange of an antisymmetric tensor in a four-point function of scalars and spin-1 operators.

In section 2.5, we specialize to the case of CFTs in four-dimensions. We develop a formalism for studying correlators of operators in arbitrary Lorentz representations using auxiliary twistor variables. Within this formalism, we define projectors and shadows for multi-twistor operators and then demonstrate their use for computing conformal blocks in a few simple examples. We conclude in section 2.6.

2.2 Constructing Conformal Blocks

2.2.1 Defining Properties

A conformal block encodes the contribution of a single irreducible conformal multiplet (a primary operator and its descendants) to a four-point function of primary operators. Consider, for example, a four-point function of primary scalars ϕ_i with dimensions Δ_i . We can expand it as a sum over conformal multiplets by inserting a complete set of states,

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4) \rangle = \sum_{\mathcal{O} \in \phi_1 \times \phi_2} \sum_{\alpha = \mathcal{O}, P\mathcal{O}, \dots} \langle 0 | \phi_3(x_3)\phi_4(x_4) | \alpha \rangle \langle \alpha | \phi_1(x_1)\phi_2(x_2) | 0 \rangle. \quad (2.3)$$

Here, $\mathcal{O} \in \phi_1 \times \phi_2$ runs over *primary* operators \mathcal{O} appearing in the OPE of $\phi_1 \times \phi_2$, and α runs over \mathcal{O} and its (normalized) descendants, considered as states in radial quantization on a sphere separating x_1, x_2 from x_3, x_4 .

For fixed \mathcal{O} , the quantities $\langle \alpha | \phi_i(x_i)\phi_j(x_j) | 0 \rangle$ are proportional to the three-point function coefficient $\lambda_{\phi_i\phi_j\mathcal{O}}$. Stripping these off, we are left with a purely kinematical quantity called a

conformal partial wave,

$$W_{\mathcal{O}}(x_i) = \frac{1}{\lambda_{\phi_1\phi_2\mathcal{O}}\lambda_{\phi_3\phi_4\mathcal{O}}} \sum_{\alpha=\mathcal{O},P\mathcal{O},\dots} \langle 0|\phi_3(x_3)\phi_4(x_4)|\alpha\rangle \langle \alpha|\phi_1(x_1)\phi_2(x_2)|0\rangle. \quad (2.4)$$

The conformal block $g_{\mathcal{O}}(u, v)$ is defined in terms of $W_{\mathcal{O}}(x_i)$ by additionally removing factors of x_{ij}^2 to obtain a dimensionless quantity,

$$W_{\mathcal{O}}(x_i) = \left(\frac{x_{14}^2}{x_{13}^2}\right)^{\frac{\Delta_{34}}{2}} \left(\frac{x_{24}^2}{x_{14}^2}\right)^{\frac{\Delta_{12}}{2}} \frac{g_{\mathcal{O}}(u, v)}{x_{12}^{\Delta_1+\Delta_2} x_{34}^{\Delta_3+\Delta_4}}, \quad (2.5)$$

where $\Delta_{ij} \equiv \Delta_i - \Delta_j$, and $u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}$, $v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$ are conformally-invariant cross-ratios.

The form of $g_{\mathcal{O}}(u, v)$ is completely fixed by conformal symmetry, and depends only on the representations of \mathcal{O} and the ϕ_i under the conformal group (i.e. their dimensions and spins). One way to see why is to note that $g_{\mathcal{O}}(u, v)$ possesses the following three properties:

1. It is invariant under conformal transformations.
2. It is an eigenvector of the quadratic Casimir of the conformal group acting on x_1, x_2 . Specifically, let L_A , with $A = 1, \dots, (d+1)(d+2)/2$ indexing the adjoint of $\text{SO}(d+1, 1)$, be generators of conformal transformations, and denote the associated differential operators acting on $\phi_i(x_i)$ by \mathcal{L}_{iA} : $\mathcal{L}_{iA}\phi_i(x_i) = [\phi_i(x_i), L_A]$. Each descendant $|\alpha\rangle$ is an eigenvector of $L^A L_A$ with the same eigenvalue $C_{\mathcal{O}} = \Delta(d - \Delta) + C_L$, where Δ is the dimension of \mathcal{O} , and C_L is the Casimir of the Lorentz representation of \mathcal{O} . Thus,

$$\begin{aligned} & (\mathcal{L}_{1A} + \mathcal{L}_{2A})(\mathcal{L}_1^A + \mathcal{L}_2^A)W_{\mathcal{O}}(x_i) \\ &= \frac{1}{\lambda_{\phi_1\phi_2\mathcal{O}}\lambda_{\phi_3\phi_4\mathcal{O}}} \sum_{\alpha} \langle 0|\phi_3(x_3)\phi_4(x_4)|\alpha\rangle \langle \alpha|[[\phi_1(x_1)\phi_2(x_2), L_A], L^A]|0\rangle \\ &= C_{\mathcal{O}}W_{\mathcal{O}}. \end{aligned} \quad (2.6)$$

Eqs. (2.6) and (2.5) then imply an eigenvalue equation for $g_{\mathcal{O}}(u, v)$.

3. The behavior of $g_{\mathcal{O}}(u, v)$ as $x_{12} \rightarrow 0$ is dictated by the primary term $\mathcal{O} \in \phi \times \phi$ in the OPE.

More explicitly, if \mathcal{O} is a spin- ℓ operator, we have

$$\begin{aligned} \phi_1(x_1)\phi_2(x_2) = & \lambda_{\phi_1\phi_2\mathcal{O}} x_{12}^{\Delta-\Delta_1-\Delta_2-\ell} x_{12\mu_1} \dots x_{12\mu_\ell} \mathcal{O}^{\mu_1\dots\mu_\ell}(x_2) \\ & + \text{descendants} + \text{other multiplets}. \end{aligned} \quad (2.7)$$

Descendants of \mathcal{O} come with higher powers of x_{12} in the OPE, and other multiplets don not contribute to $g_{\mathcal{O}}$. Hence the small x_{12} limit of our conformal block comes from the leading term above,

$$g_{\mathcal{O}}(u, v) \sim x_{12}^{\Delta-\ell} x_{12\mu_1} \dots x_{12\mu_\ell} \langle \mathcal{O}^{\mu_1\dots\mu_\ell}(x_2) \phi_3(x_3) \phi_4(x_4) \rangle. \quad (2.8)$$

Together, these properties determine $g_{\mathcal{O}}(u, v)$. This is demonstrated for example in [65] where Dolan and Osborn explicitly solve (2.6) subject to (2.8). In even dimensions, their solution takes a simple form in terms of hypergeometric functions. For instance when $d = 4$,

$$\begin{aligned} g_{\mathcal{O}}(u, v) = & (-1)^\ell \frac{z\bar{z}}{z-\bar{z}} (k_{\Delta+\ell}(z) k_{\Delta-\ell-2}(\bar{z}) - z \leftrightarrow \bar{z}) \\ k_\beta(x) \equiv & x^{\beta/2} {}_2F_1\left(\frac{\beta-\Delta_{12}}{2}, \frac{\beta+\Delta_{34}}{2}, \beta; x\right), \end{aligned} \quad (2.9)$$

where Δ, ℓ are the dimension and spin of \mathcal{O} , respectively, and z and \bar{z} are defined in terms of the cross ratios u and v by

$$u = z\bar{z}, \quad v = (1-z)(1-\bar{z}). \quad (2.10)$$

In more general situations, the conformal Casimir equation becomes a complicated system of coupled PDEs that can be difficult to solve. Instead of solving it directly, our approach will be to write down expressions that manifestly satisfy properties 1, 2, and 3, and then compute them. This method, essentially the shadow formalism [48–51], was used in Dolan and Osborn’s original derivation of (2.9) [64]. Our contribution will be to clarify and generalize this approach, providing a unified way to ensure each of the above properties holds, along with a toolkit for performing the resulting calculations. To this end, let us address each property in turn.

2.2.2 The Embedding Space

The constraints of conformal invariance are most transparent in the embedding space [47, 70–73]. Consider a Euclidean CFT in d -dimensions, with conformal group $\text{SO}(d+1, 1)$ acting nonlinearly on spacetime \mathbb{R}^d . The key idea, originally due to Dirac, is that this nonlinear action is induced from the much simpler linear action of $\text{SO}(d+1, 1)$ on the “embedding space” $\mathbb{R}^{d+1,1}$. To see how, choose coordinates $X^m = (X^+, X^-, X^\mu)$ on $\mathbb{R}^{d+1,1}$, with the inner product

$$X \cdot X = \eta_{mn} X^m X^n = -X^+ X^- + X_\mu X^\mu. \quad (2.11)$$

The condition $X^2 = 0$ defines an $\text{SO}(d+1, 1)$ -invariant subspace of dimension $d+1$ — the null-cone. We obtain d -dimensional Euclidean space by projectivizing: quotienting the null-cone by the rescaling $X \sim \lambda X$, $\lambda \in \mathbb{R}$. Because projectivizing respects Lorentz rotations of the embedding space, the projective null-cone naturally inherits an action of $\text{SO}(d+1, 1)$.

We can identify the projective null-cone with \mathbb{R}^d by “gauge-fixing” this rescaling. For example, imposing the gauge condition $X^+ = 1$, null vectors take the form $X = (1, x^2, x^\mu)$, for $x^\mu \in \mathbb{R}^d$.¹ This gauge slice is called the *Poincaré section*. Beginning with some point $X = (1, x^2, x^\mu)$, a transformation $h \in \text{SO}(d+1, 1)$ takes X to hX by matrix multiplication. To get back to the Poincaré section, we must further rescale $hX \rightarrow hX/(hX)^+$. The combined transformation $X \rightarrow hX/(hX)^+$ is precisely the nonlinear action of the conformal group on \mathbb{R}^d . Note that on the Poincaré section, we have

$$-2X \cdot Y = (x - y)^2. \quad (2.12)$$

Primary operators on \mathbb{R}^d can be lifted to homogeneous, conformally-covariant fields on the null-cone. For example, given a primary scalar $\phi(x)$ with dimension Δ , one can define a scalar

¹This gauge condition fails for precisely one null direction, $X = (0, 1, 0)$ representing the point at infinity.

on the entire null-cone by

$$\Phi(X) \equiv (X^+)^{-\Delta} \phi(X^\mu/X^+). \quad (2.13)$$

The field $\Phi(X)$ then transforms simply under conformal transformations $\Phi(X) \rightarrow \Phi(hX)$. Conformal invariance means that correlators of $\Phi(X)$ are invariant under linear $\text{SO}(d+1, d)$ rotations.

The dimension of ϕ is reflected in the degree of Φ ,

$$\Phi(\lambda X) = \lambda^{-\Delta} \Phi(X). \quad (2.14)$$

This homogeneity condition must be respected by any correlator involving $\Phi(X)$. For example, the two-point function $\langle \Phi(X_1) \Phi(X_2) \rangle$ is fixed by conformal invariance, homogeneity, and the null condition $X_i^2 = 0$ to have the form

$$\langle \Phi(X_1) \Phi(X_2) \rangle \propto \frac{1}{X_{12}^\Delta}, \quad X_{ij} \equiv -2X_i \cdot X_j. \quad (2.15)$$

The notation X_{ij} is for convenience when comparing to flat-space coordinates on the Poincaré section, $X_{ij} \rightarrow x_{ij}^2$. In our conventions, Φ is canonically normalized when the constant of proportionality in (2.15) is 1.

One can additionally lift fields with spin to conformally covariant fields on the null-cone [68]. We defer discussion of this machinery until it is needed in section 2.4.2. In what follows, we will write simply $\phi(X)$ to indicate the lift of $\phi(x)$ to the embedding space.

2.2.3 Conformal Integrals

The projective null-cone admits a natural notion of integration that produces new conformal invariants from old ones. Let us start with an obvious $\text{SO}(d+1, 1)$ -invariant measure on the null-cone, $d^{d+2}X \delta(X^2)$, where $\delta(X^2)$ is a Dirac delta-function. This measure has degree d in X , so only its product with a degree $-d$ function $f(X)$ is well-defined after projectivization. However

the integral

$$\int d^{d+2}X \delta(X^2) f(X) \quad (2.16)$$

is formally infinite because of the rescaling invariance $X \rightarrow \lambda X$.

We can obtain a finite result by dividing by the volume of the “gauge-group,”²

$$\int D^d X f(X) \equiv \frac{1}{\text{Vol GL}(1, \mathbb{R})^+} \int_{X^+ + X^- \geq 0} d^{d+2}X \delta(X^2) f(X). \quad (2.17)$$

Integrals of this form, which we call “conformal integrals,” will play a central role in this work.³

In practice, we can evaluate them by gauge-fixing and supplying the appropriate Faddeev-Popov determinant. For example, the gauge choice $X^+ = 1$ reduces (2.17) to a conventional integral over flat space. The advantage of the definition (2.17) is that it makes $\text{SO}(d+1, 1)$ -invariance manifest.

As an example, let us evaluate a conformal integral which will be important in subsequent computations,

$$I(Y) = \int D^d X \frac{1}{(-2X \cdot Y)^d} \quad (Y^2 < 0). \quad (2.18)$$

Note that this is essentially the unique conformal integral depending on a single vector Y and producing a scalar. The requirement that the integrand have degree $-d$ in X , along with the null condition $X^2 = 0$ fixes the integrand up to a constant.

Since $I(Y)$ is conformally invariant and homogeneous in Y , we are free to choose $Y = Y_0 = (1, 1, 0)$ with $Y_0^2 = -1$, and recover the full Y -dependence at the end from dimensional analysis.

²Precisely, we quotient by the connected component of the identity $\text{GL}(1, \mathbb{R})^+ \subset \text{GL}(1, \mathbb{R})$ and restrict the integral to a single branch of the null cone.

³An alternative definition of the conformal integral measure is as a residue $D^d X = \frac{1}{2\pi i} \oint_{S^1} \frac{\omega}{X^2}$, where $\omega = \frac{1}{(d+1)!} \epsilon_{m_0 \dots m_{d+1}} X^{m_0} dX^{m_1} \wedge \dots \wedge dX^{m_{d+1}}$ is an $\text{SO}(d+2)$ -invariant volume form on projective space \mathbb{P}^{d+1} , and the S^1 encircles the locus where $X^2 = 0$. The combination $\frac{\omega}{X^2}$ has projective weight d , so it can be integrated against a section with projective weight $-d$. The full integration contour we consider has topology $S^1 \times S^d$.

From the definition of the measure (2.17), we have

$$\begin{aligned}
I(Y_0) &= \frac{1}{\text{Vol GL}(1, \mathbb{R})^+} \int_{X^+ + X^- \geq 0} d^{d+2} X \delta(-X^+ X^- + X_\mu X^\mu) \frac{1}{(X^+ + X^-)^d} \\
&= \frac{1}{\text{Vol GL}(1, \mathbb{R})^+} \int d^d X \int_0^\infty \frac{dX^+}{X^+} \frac{1}{(X^+ + X_\mu X^\mu / X^+)^d} \\
&= \int d^d X \frac{1}{(1 + X_\mu X^\mu)^d} = \frac{\pi^{d/2} \Gamma(d/2)}{\Gamma(d)}. \tag{2.19}
\end{aligned}$$

In the third line, we have made the gauge choice $X^+ = 1$. The associated Faddeev-Popov determinant is 1. Restoring the factors of $-Y^2$ required by dimensional analysis gives

$$I(Y) = \frac{\pi^{d/2} \Gamma(d/2)}{\Gamma(d)} \frac{1}{(-Y^2)^{d/2}}. \tag{2.20}$$

Eq. (2.20) is sufficient for evaluating numerous conformal integrals. For instance, products $\prod_i (-2X \cdot Y_i)^{-a_i}$ can be reduced to the form (2.18) using the Feynman/Schwinger parameterization

$$\frac{1}{\prod_i A_i^{a_i}} = \frac{\Gamma(\sum_i a_i)}{\prod_i \Gamma(a_i)} \int_0^\infty \prod_{i=2}^n d\alpha_i \alpha_i^{a_i-1} \frac{1}{(A_1 + \sum_{i=2}^n \alpha_i A_i)^{\sum_i a_i}}. \tag{2.21}$$

Combining (2.21) and (2.20), a three-point integral is given by

$$\int D^d X \frac{1}{X_{10}^a X_{20}^b X_{30}^c} = \frac{\pi^h \Gamma(h-a) \Gamma(h-b) \Gamma(h-c)}{\Gamma(a) \Gamma(b) \Gamma(c)} \frac{1}{X_{12}^{h-c} X_{13}^{h-b} X_{23}^{h-a}}, \tag{2.22}$$

where $h \equiv d/2$ and $a + b + c = d$ so that the projective measure is well-defined. Note that the form of this result is fixed by homogeneity in X_1, X_2, X_3 .

More generally, any conformal integral can be manipulated to a sum of terms of the form

$$\begin{aligned}
\int D^d X \frac{X^{m_1} \dots X^{m_n}}{(-2X \cdot Y)^{d+n}} &= \frac{\Gamma(d)}{2^n \Gamma(d+n)} \left(\prod_i \frac{\partial}{\partial Y_{m_i}} \right) I(Y) \\
&= \frac{\pi^{d/2} \Gamma(d/2 + n)}{\Gamma(d+n)} \frac{Y^{m_1} \dots Y^{m_n}}{(-Y^2)^{d/2+n}} - \text{traces}, \tag{2.23}
\end{aligned}$$

where traces are subtracted using the embedding space metric η_{mn} . Tracelessness is clear in the integrand because X^2 vanishes on the null-cone. Eqs (2.20) and (2.23) undergird most of the computations in this work.

2.2.4 The Conformal Casimir

Three-point functions of primary operators provide natural eigenvectors of the conformal Casimir.⁴ Because $\langle\phi_1\phi_2\mathcal{O}\rangle$ is conformally covariant, we have

$$(\mathcal{L}_{1A} + \mathcal{L}_{2A})\langle\phi_1(X_1)\phi_2(X_2)\mathcal{O}(X_3)\rangle = -\mathcal{L}_{3A}\langle\phi_1(X_1)\phi_2(X_2)\mathcal{O}(X_3)\rangle. \quad (2.24)$$

Thus, action of the conformal Casimir on X_1, X_2 is equivalent to action on X_3 , which gives simply the eigenvalue $C_{\mathcal{O}}$,

$$\begin{aligned} (\mathcal{L}_{1A} + \mathcal{L}_{2A})(\mathcal{L}^{1A} + \mathcal{L}^{2A})\langle\phi_1(X_1)\phi_2(X_2)\mathcal{O}(X_3)\rangle &= \mathcal{L}^{3A}\mathcal{L}_{3A}\langle\phi_1(X_1)\phi_2(X_2)\mathcal{O}(X_3)\rangle \\ &= C_{\mathcal{O}}\langle\phi_1(X_1)\phi_2(X_2)\mathcal{O}(X_3)\rangle. \end{aligned} \quad (2.25)$$

This argument is independent of the actual value of X_3 , so any linear combination of $\langle\phi_1(X_1)\phi_2(X_2)\mathcal{O}(X_3)\rangle$ with different values of X_3 is also an eigenvector of the conformal Casimir acting on X_1, X_2 , with the same eigenvalue. In particular, so is the conformal integral

$$\int D^d X_3 \langle\phi_1(X_1)\phi_2(X_2)\mathcal{O}(X_3)\rangle f(X_3) \quad (2.26)$$

where $f(X)$ is any homogeneous function on the null-cone with degree $\Delta - d$.

This suggests a natural candidate for the conformal partial wave $W_{\mathcal{O}}$,

$$W_{\mathcal{O}}(X_i) \stackrel{?}{=} \frac{1}{\mathcal{N}_{\mathcal{O}}} \int D^d X D^d Y \langle\phi_1(X_1)\phi_2(X_2)\mathcal{O}(X)\rangle \frac{1}{(-2X \cdot Y)^{d-\Delta}} \langle\mathcal{O}(Y)\phi_3(X_3)\phi_4(X_4)\rangle, \quad (2.27)$$

where $\mathcal{N}_{\mathcal{O}}$ is a constant to be determined. Note that (2.27) has the correct degree in the X_i , is manifestly conformally invariant, and is also manifestly an eigenvector of the conformal Casimir acting on X_1, X_2 (equivalently X_3, X_4) with the correct eigenvalue, since it has the form (2.26). The denominator $(-2X \cdot Y)^{d-\Delta}$ is the unique choice for which the conformal integrals over X and

⁴Note that the differential operators generating conformal transformations in the embedding space are just the usual generators of $\text{SO}(d+1, 1)$ acting on functions on $\mathbb{R}^{d+1, 1}$, $\mathcal{L}_{mn} = X_m \frac{\partial}{\partial X^n} - X_n \frac{\partial}{\partial X^m}$.

Y are well-defined. We will see shortly that (2.27) is incorrect, but is a convenient stepping stone to the correct answer.

We can rewrite (2.27) in a useful way by introducing the *shadow operator*,

$$\tilde{\mathcal{O}}(X) = \int D^d Y \frac{1}{(-2X \cdot Y)^{d-\Delta}} \mathcal{O}(Y), \quad (2.28)$$

which formally has the transformation properties of a primary scalar with dimension $d - \Delta$.⁵ Note that $\tilde{\mathcal{O}}$ has the same eigenvalue as \mathcal{O} under the conformal Casimir, since $C_{\mathcal{O}}$ is invariant under $\Delta \rightarrow d - \Delta$. In terms of $\tilde{\mathcal{O}}$, eq. (2.27) reads

$$W_{\mathcal{O}}(X_i) \stackrel{?}{=} \frac{1}{\mathcal{N}_{\mathcal{O}}} \int D^d X_0 \langle \phi_1(X_1) \phi_2(X_2) \mathcal{O}(X_0) \rangle \langle \tilde{\mathcal{O}}(X_0) \phi_3(X_3) \phi_4(X_4) \rangle. \quad (2.29)$$

For example, in the special case where the Δ_i are all equal to δ , we have the candidate conformal block

$$g_{\mathcal{O}}(X_i) \stackrel{?}{=} \frac{1}{\mathcal{N}_{\mathcal{O}}} \frac{\Gamma(\Delta) \Gamma^2(\frac{d-\Delta}{2})}{\pi^{d/2} \Gamma(d-\Delta) \Gamma^2(\frac{\Delta}{2})} \int D^d X_0 \frac{X_{12}^{\Delta/2} X_{34}^{(d-\Delta)/2}}{X_{10}^{\Delta/2} X_{20}^{\Delta/2} X_{03}^{(d-\Delta)/2} X_{04}^{(d-\Delta)/2}} \equiv F(X_i), \quad (2.30)$$

where again $X_{ij} \equiv -2X_i \cdot X_j$, and we have evaluated $\langle \tilde{\mathcal{O}}(X_0) \phi_3(X_3) \phi_4(X_4) \rangle$ using (2.22).

2.2.5 Consistency with the OPE

Eq. (2.30) is a conformally-invariant eigenvector of the conformal Casimir with the correct eigenvalue. Our final requirement is that it have the correct limiting behavior as $X_{12} \rightarrow 0$, namely $g_{\mathcal{O}}(X_i) \sim X_{12}^{\Delta/2}$. This is indeed the behavior of the *integrand* above. But the full behavior of the *integral* $F(X_i)$ is unclear. The integral over X_0 could potentially probe the region near X_1, X_2 in ways that introduce new singularities.

In fact this *must* happen, since we could have performed the X integral in (2.27) first, exchanging $\Delta \leftrightarrow d - \Delta$ in the integrand of (2.30). Symmetry under $\Delta \leftrightarrow d - \Delta$ implies that $F(X_i)$ must actually compute a linear combination of the conformal block $g_{\mathcal{O}}$ and its shadow block

⁵Since $\tilde{\mathcal{O}}$ is nonlocal, this does not contradict unitarity.

$g_{\tilde{\mathcal{O}}}$ (which has the same eigenvalue under the conformal Casimir, but different limiting behavior $g_{\tilde{\mathcal{O}}}(X_i) \sim X_{12}^{(d-\Delta)/2}$ as $X_{12} \rightarrow 0$). In other words,

$$F(X_i) = g_{\mathcal{O}}(X_i) + K_{\mathcal{O}} g_{\tilde{\mathcal{O}}}(X_i), \quad (2.31)$$

where $K_{\mathcal{O}}$ is a constant.

Thus, our final step should be to remove the shadow component $g_{\tilde{\mathcal{O}}}(X_i)$ from $F(X_i)$. This procedure can be performed quickly and elegantly in Mellin space [77], but takes some care in position space. The approach of [64] is to evaluate integrals like (2.30) as a series in conformal cross ratios $u, 1-v$, discard terms of the form $u^{(d-\Delta)/2+n}(1-v)^m$, $m, n \in \mathbb{Z}$, which belong to the shadow block, and re-sum the remaining terms.

Here, we will take a cleaner approach that avoids complicated series expansions and special function identities. The key observation is that $g_{\mathcal{O}}$ and $g_{\tilde{\mathcal{O}}}$ are distinguished by their behavior under monodromy $M : X_{12} \rightarrow e^{4\pi i} X_{12}$,⁶

$$M : g_{\mathcal{O}} \rightarrow e^{2\pi i \Delta} g_{\mathcal{O}} \quad (2.32)$$

$$M : g_{\tilde{\mathcal{O}}} \rightarrow e^{2\pi i (d-\Delta)} g_{\tilde{\mathcal{O}}}. \quad (2.33)$$

Isolating $g_{\mathcal{O}}$ means projecting $F(X_i)$ onto the correct eigenspace of M ,

$$g_{\mathcal{O}}(X_i) = F(X_i)|_{M=e^{2\pi i \Delta}}. \quad (2.34)$$

Since M commutes with conformal transformations, so does projection onto its eigenspaces. Consequently, (2.34) is still conformally invariant, and still solves the correct Casimir differential equation. Thus, it satisfies the requirements for a conformal block, and all that remains is to compute it. We will do so in section 2.3. We give more detail about how monodromy projection ensures the correct small x_{12} behavior and why the shadow block $g_{\tilde{\mathcal{O}}}$ appears in appendix 2.A.

⁶ M can be generated by exponentiating a dilatation operator $e^{2\pi i (\mathcal{D}_1 + \mathcal{D}_2)}$ acting on X_1, X_2 . See appendix 2.A for details.

2.2.6 Projectors and Shadows

Our prescription (2.29) for computing conformal blocks can be summarized succinctly as the statement that

$$|\mathcal{O}| \equiv \frac{1}{\mathcal{N}_{\mathcal{O}}} \int D^d X |\mathcal{O}(X)\rangle \langle \tilde{\mathcal{O}}(X)| \quad (2.35)$$

is a projector onto the conformal multiplet of \mathcal{O} . The object $|\mathcal{O}|$, inserted within a correlator $\langle \phi_1 \dots \phi_m \phi_{m+1} \dots \phi_n \rangle$, is shorthand for the conformal integral of a product of correlators, supplemented by appropriate monodromy projections

$$\langle \phi_1 \dots \phi_m | \mathcal{O} | \phi_{m+1} \dots \phi_n \rangle \equiv \frac{1}{\mathcal{N}_{\mathcal{O}}} \int D^d X \langle \phi_1 \dots \phi_m \mathcal{O}(X) \rangle \langle \tilde{\mathcal{O}}(X) \phi_{m+1} \dots \phi_n \rangle \Big|_{M=e^{2\pi i \varphi}}. \quad (2.36)$$

Here M maps $X_{ij} \rightarrow e^{4\pi i} X_{ij}$ for $i, j \leq m$, and leaves the other X_{ij} invariant. Consistency with the OPE requires $\varphi = \Delta - \sum_{i \leq m} \Delta_i$. The notion of $|\mathcal{O}|$ as a projection operator is somewhat formal, since the precise form of the monodromy projection depends on what correlator we are computing.

The constant $\mathcal{N}_{\mathcal{O}}$ can be fixed by demanding that $|\mathcal{O}|$ act trivially when inserted within a correlator involving \mathcal{O} ,

$$\langle \mathcal{O}(X) | \mathcal{O} | \dots \rangle = \frac{1}{\mathcal{N}_{\mathcal{O}}} \int D^d X_1 \langle \mathcal{O}(X) \mathcal{O}(X_1) \rangle \frac{D^d X_0}{(-2X_1 \cdot X_0)^{d-\Delta}} \langle \mathcal{O}(X_0) \dots \rangle \quad (2.37)$$

$$= \frac{1}{\mathcal{N}_{\mathcal{O}}} \int \frac{D^d X_1}{(-2X \cdot X_1)^{\Delta}} \frac{D^d X_0}{(-2X_1 \cdot X_0)^{d-\Delta}} \langle \mathcal{O}(X_0) \dots \rangle \quad (2.38)$$

$$\stackrel{?}{=} \langle \mathcal{O}(X) \dots \rangle. \quad (2.39)$$

Fortunately, we can determine $\mathcal{N}_{\mathcal{O}}$ from this condition without too much computation. Note that any correlator $\langle \mathcal{O}(X_0) \dots \rangle$ can be written as a linear combination of functions

$$\frac{1}{(-2X_0 \cdot Y)^{\Delta}}, \quad (2.40)$$

where Y is some (not necessarily null) vector. For instance, we may combine denominators using Feynman parameters, so that Y is a combination of parameters and other points in the correlator.⁷

⁷Feynman parameterization is singular for numerator factors with positive integer powers $(-2X \cdot X_i)^n$. For the argument here, one should regulate these singularities by taking $n \rightarrow n + \epsilon$.

The power of X_0 is fixed by homogeneity. We have

$$\int \frac{D^d X_0}{(-2X_1 \cdot X_0)^{d-\Delta}} \frac{1}{(-2X_0 \cdot Y)^\Delta} = \frac{\pi^h \Gamma(\Delta - h)}{\Gamma(\Delta)} \frac{(-Y^2)^{h-\Delta}}{(-2X_1 \cdot Y)^{d-\Delta}}, \quad (2.41)$$

where $h \equiv d/2$.⁸ Iterating this formula a second time with $\Delta \rightarrow d - \Delta$ gives

$$\int \frac{D^d X_1}{(-2X_2 \cdot X_1)^\Delta} \frac{D^d X_0}{(-2X_1 \cdot X_0)^{d-\Delta}} \frac{1}{(-2X_0 \cdot Y)^\Delta} = \frac{\pi^d \Gamma(\Delta - h) \Gamma(h - \Delta)}{\Gamma(\Delta) \Gamma(d - \Delta)} \frac{1}{(-2X_2 \cdot Y)^\Delta}. \quad (2.42)$$

This result has exactly the same form as our starting point (2.40), up to a Y -independent constant.

Taking linear combinations for different Y , it follows that

$$\langle \mathcal{O}(X) | \mathcal{O} | \dots \rangle = \frac{1}{\mathcal{N}_{\mathcal{O}}} \frac{\pi^d \Gamma(\Delta - h) \Gamma(h - \Delta)}{\Gamma(\Delta) \Gamma(d - \Delta)} \langle \mathcal{O}(X) \dots \rangle, \quad (2.43)$$

so we should choose

$$\mathcal{N}_{\mathcal{O}} = \frac{\Gamma(\Delta) \Gamma(d - \Delta)}{\pi^d \Gamma(\Delta - h) \Gamma(h - \Delta)}. \quad (2.44)$$

Our strategy for computing higher-spin conformal blocks will be to find conformally-invariant projectors analogous to (2.35) for operators in nontrivial Lorentz representations. Inserting the projector within a four-point function, we obtain expressions for conformal partial waves in terms of monodromy-projected conformal integrals. We give further details in section 2.4.1. For now, let us turn to actually computing those integrals.

2.3 Conformal Integrals and Monodromy Invariants

2.3.1 Scalar Four-point Integrals

As we saw in the previous section, the conformal block for scalar exchange in a four-point function $\langle \phi \phi \phi \phi \rangle$ depends on the monodromy-projected conformal integral

$$F(X_i)|_{M=e^{2\pi i \Delta}} \propto \int D^d X_0 \frac{1}{X_{10}^{\Delta/2} X_{20}^{\Delta/2} X_{03}^{(d-\Delta)/2} X_{04}^{(d-\Delta)/2}} \Big|_{M=1} \times X_{12}^{\Delta/2} X_{34}^{(d-\Delta)/2}, \quad (2.45)$$

⁸For $|\mathcal{O}|$ inserted within a two-point function $\langle \mathcal{O}(X) | \mathcal{O} | \mathcal{O}(Y) \rangle$, the vector Y is null and this intermediate result is singular. Taking Y slightly off the null-cone provides a regularization.

where $M : X_{12} \rightarrow e^{4\pi i} X_{12}$. In this section, we will give a simple computation of the above quantity and its generalizations.⁹ In even dimensions, the result can be cast in terms of elementary hypergeometric functions, reproducing expressions in [64, 65]. Unlike the derivation in [64, 65], ours easily generalizes to the case of conformal integrals with tensor indices, which will be needed to compute conformal blocks for operators with spin.

Notice that removing $X_{12}^{\Delta/2}$ from the integrand in (2.45) leaves an integral projected onto its monodromy-invariant subspace. This occurs in more general computations, so let us compute the $M = 1$ projection of the four-point integral

$$I(X_i) = \frac{\Gamma(a)\Gamma(b)\Gamma(c)\Gamma(d)}{\pi^h \Gamma(h)} \int \frac{D^{2h} X_0}{X_{10}^a X_{20}^b X_{03}^c X_{04}^d}, \quad (2.46)$$

where $d = 2h$ is the dimension of spacetime, and $a + b + c + d = 2h$ so that the projective measure is well-defined. The constants out front are chosen for later convenience. We will assume $X_{ij} > 0$.

To begin, combine denominators with the Feynman/Schwinger parameterization (2.21) and apply (2.20) to obtain

$$I(X_i) = \int_0^\infty \frac{d\beta}{\beta} \frac{d\gamma}{\gamma} \frac{d\delta}{\delta} \frac{\beta^b \gamma^c \delta^d}{(-(X_1 + \beta X_2 + \gamma X_3 + \delta X_4)^2)^h} \equiv I_{b,e,f}^{(h)}(X_i) \quad (2.47)$$

$$= \frac{\Gamma(h-f)\Gamma(f)}{\Gamma(h)} \int_0^\infty \frac{d\beta}{\beta} \frac{d\gamma}{\gamma} \frac{\beta^b \gamma^c}{(\beta X_{12} + \gamma X_{13} + \beta \gamma X_{23})^{h-f} (X_{14} + \beta X_{24} + \gamma X_{34})^f}, \quad (2.48)$$

where in the last line we have performed the integral over δ . We denote the integral (2.47) as $I_{b,e,f}^{(h)}(X_i)$ for convenience in later sections.

Let us clarify the analytic structure of (2.48). With β fixed, the integral over γ traces a path on a multi-sheeted cover $\Sigma \rightarrow \mathbb{P}^1$, with branch points at

$$0, \quad \gamma_1 \equiv -\frac{\beta X_{12}}{X_{13} + \beta X_{23}}, \quad \gamma_2 \equiv -\frac{X_{14} + \beta X_{24}}{X_{34}}, \quad \text{and } \infty. \quad (2.49)$$

⁹Restricting to the Poincaré section $X^+ = 1$, conformal 2-, 3-, and 4-point integrals become generalized bubble, triangle, and box integrals, and our results are consistent with known results in those cases. For recent computations of these integrals in Mellin space, see [76].

Note that for sufficiently small X_{12} , we have $|\gamma_1| < |\gamma_2|$, independent of the value of β . We may deform the γ contour as depicted in figure 2.1, so that it follows the negative real axis, moving above γ_1 and γ_2 . Our γ -integral (2.48) can thus be written

$$I = I_1 + I_2 + I_3, \quad (2.50)$$

where I_1, I_2, I_3 are integrals along the intervals $[0, \gamma_1]$, $[\gamma_1, \gamma_2]$, and $[\gamma_2, \infty]$, respectively.¹⁰

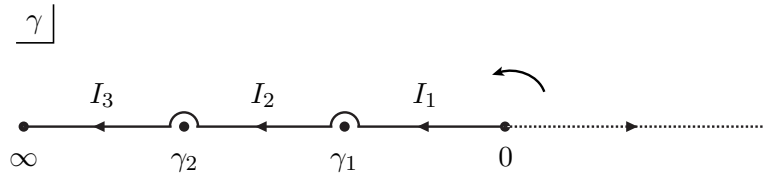


Figure 2.1: We rotate the contour for the integral (2.48) in the complex γ -plane so that it passes along the negative real axis. It breaks up into I_1, I_2, I_3 as shown.

The integrals I_1, I_2, I_3 are not linearly independent. A contour encircling all four branch points $0, \gamma_1, \gamma_2, \infty$ is contractible on Σ , so integrates to zero. On the other hand, such a contour can be deformed to a linear combination of I_1, I_2, I_3 as shown in figure 2.2, so that

$$0 = I_1(1 - e^{-i\phi_0}) + I_2(1 - e^{i\phi_\infty + i\phi_2}) + I_3(1 - e^{i\phi_\infty}), \quad (2.51)$$

where

$$\phi_0 \equiv 2\pi e, \quad \phi_1 \equiv 2\pi(f - h), \quad \phi_2 \equiv -2\pi f, \quad \phi_\infty \equiv 2\pi(h - e) \quad (2.52)$$

are the phases associated with moving counterclockwise around each branch point. From (2.51) we can solve for I_2 and express I in terms of I_1 and I_3 ,

$$I = \frac{e^{-i\phi_0} - e^{i\phi_\infty + i\phi_2}}{1 - e^{i\phi_\infty + i\phi_2}} I_1 + \frac{e^{i\phi_\infty} - e^{i\phi_\infty + i\phi_2}}{1 - e^{i\phi_\infty + i\phi_2}} I_3. \quad (2.53)$$

¹⁰These integrals may have power-law singularities γ^{-x} with non-integral x at their endpoints. We define them by analytic continuation in x .

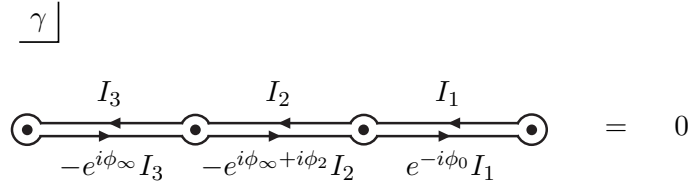


Figure 2.2: This contour is contractible on the punctured Riemann sphere, and thus integrates to zero. However, it can also be written as a linear combination of the segments I_1, I_2, I_3 as shown. The associated phases are determined by the monodromy around the branch points $0, \gamma_1, \gamma_2, \infty$.

We are finally ready to understand the behavior of $I(X_i)$ under monodromy $M : X_{12} \rightarrow e^{4\pi i} X_{12}$. M moves the branch point γ_1 twice around the origin (figure 2.3), so that the integral I_1 picks up a phase $e^{2i\phi_0}$. On the other hand, M leaves the integral I_3 invariant, since neither γ_2 nor ∞ moves, and γ_1 does not pass through the I_3 integration contour. In other words, (2.53) is precisely the decomposition of I into eigenvectors of M . The monodromy-invariant component is

$$I|_{M=1} = \frac{e^{i\phi_\infty} - e^{i\phi_\infty + i\phi_2}}{1 - e^{i\phi_\infty + i\phi_2}} I_3 = e^{i\pi(h-e)} \frac{\sin(\pi f)}{\sin(\pi(e+f-h))} I_3. \quad (2.54)$$

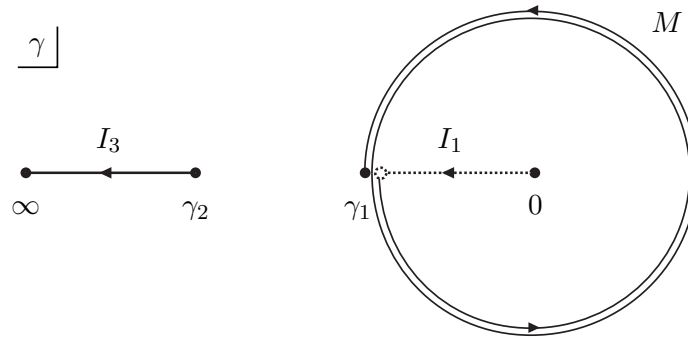


Figure 2.3: The monodromy M moves γ_1 twice around the origin, so that I_1 picks up a phase $e^{2i\phi_0}$, while I_3 remains invariant.

Having identified the correct monodromy-invariant contour, let us change variables in (2.48)

to $\beta \rightarrow \frac{X_{14}}{X_{24}}\beta$, $\gamma \rightarrow e^{i\pi \frac{X_{14}}{X_{34}}}\gamma$. This maps $\gamma_2 \rightarrow \beta + 1$ and gives

$$I|_{M=1} = \frac{\Gamma(h-f)\Gamma(f)}{\Gamma(h)} \frac{\sin(\pi f)}{\sin(\pi(e+f-h))} X_{14}^{b+e-h} X_{13}^{f-h} X_{34}^{h-f-e} X_{24}^{-b} \\ \times \int_0^\infty \frac{d\beta}{\beta} \int_{\beta+1}^\infty \frac{d\gamma}{\gamma} \frac{\beta^b \gamma^e}{(\gamma + v\beta\gamma - u\beta)^{h-f} (\gamma - \beta - 1)^f}. \quad (2.55)$$

It's now straightforward to expand the denominator and evaluate the integral as a power series in u and $1 - v$.

When the dimension of spacetime is even, so that h is an integer, we can proceed further.

The computation is easiest when the exponents in the denominator sum to 1, so let us bring (2.55) to this form:

$$I|_{M=1} = \frac{\pi}{\Gamma(h) \sin(\pi(e+f-h))} X_{14}^{b+e-h} X_{13}^{f-h} X_{34}^{h-f-e} X_{24}^{-b} \\ \times \left(-\frac{\partial}{\partial v} \right)^{h-1} \int_0^\infty \frac{d\beta}{\beta} \int_{\beta+1}^\infty \frac{d\gamma}{\gamma} \frac{\beta^{b-h+1} \gamma^{e-h+1}}{(\gamma + v\beta\gamma - u\beta)^{1-f} (\gamma - \beta - 1)^f} \quad (h \in \mathbb{Z}). \quad (2.56)$$

Finally, write $u = z\bar{z}$, $v = (1-z)(1-\bar{z})$, and make the change of variables

$$\beta = \frac{s}{(1-s)(1-t\bar{z})}, \quad \gamma = \frac{1}{(1-t)(1-s)}, \quad s, t \in [0, 1]. \quad (2.57)$$

Our expression factorizes into a product of one-dimensional integrals which produce ${}_2F_1$ hypergeometric functions of z and \bar{z} ,

$$I|_{M=1} = \frac{\Gamma(2h-b-e-f)\Gamma(1+b-h)\Gamma(1-f)\Gamma(h-e)\Gamma(e+f-h)}{\Gamma(h)\Gamma(1+h-e-f)} X_{14}^{b+e-h} X_{13}^{f-h} X_{34}^{h-f-e} X_{24}^{-b} \\ \times \left(-\frac{\partial}{\partial v} \right)^{h-1} F(z)F(\bar{z}) \quad (\text{even dimensions, } h \in \mathbb{Z}) \\ F(x) \equiv {}_2F_1(b+1-h, 1-f, 1+h-e-f, x). \quad (2.58)$$

In terms of z and \bar{z} , the derivative operator reads

$$-\frac{\partial}{\partial v} = \frac{1}{z-\bar{z}} \left(z \frac{\partial}{\partial z} - \bar{z} \frac{\partial}{\partial \bar{z}} \right). \quad (2.59)$$

When $h \notin \mathbb{Z}$, the change of variables (2.57) does not factorize the integral (2.55), but instead gives a result which can be expanded as a series of hypergeometric functions.

With (2.58) we can give compact expressions for scalar conformal blocks in even dimensions. The monodromy projection of the right hand side of (2.29) gives the conformal partial wave for exchange of a scalar \mathcal{O} with dimension Δ between scalars ϕ_i with dimensions Δ_i . The conformal block is given by

$$g_{\mathcal{O}}(X_i) = X_{12}^{\frac{\Delta_1+\Delta_2}{2}} X_{34}^{\frac{\Delta_3+\Delta_4}{2}} \left(\frac{X_{13}}{X_{14}} \right)^{\frac{\Delta_{34}}{2}} \left(\frac{X_{14}}{X_{24}} \right)^{\frac{\Delta_{12}}{2}} \times \int D^d X_0 \langle \phi_1(X_1) \phi_2(X_2) \mathcal{O}(X_0) \rangle \langle \tilde{\mathcal{O}}(X_0) \phi_3(X_3) \phi_4(X_4) \rangle \Big|_{M=e^{2\pi i \Delta}} \quad (2.60)$$

$$= \frac{\left(\frac{\Delta-\Delta_{12}}{2} \right)_{1-h} \left(\frac{\Delta+\Delta_{34}}{2} \right)_{1-h}}{(\Delta)_{1-h}} \left(-\frac{\partial}{\partial v} \right)^{h-1} k(z) k(\bar{z})$$

$$k(x) \equiv x^{\Delta/2} {}_2F_1 \left(\frac{\Delta-\Delta_{12}}{2} + 1 - h, \frac{\Delta+\Delta_{34}}{2} + 1 - h, \Delta + 1 - h, x \right), \quad (2.61)$$

where $\Delta_{ij} \equiv \Delta_i - \Delta_j$, and $(a)_n = \Gamma(a+n)/\Gamma(a)$ is the Pochhammer symbol. This agrees with the results of [64, 65] for 2, 4, and 6 dimensions, after applying elementary hypergeometric function identities. In particular, eq. (2.9) for $d=4$, $\ell=0$ is easily verified.

2.3.2 Tensor Four-point Integrals

These results generalize straightforwardly to conformal integrals with nontrivial tensor structure. The most general possible four-point integral is

$$I^{m_1 \dots m_n}(X_i) = \frac{\Gamma(a)\Gamma(b)\Gamma(e)\Gamma(f)}{\pi^h \Gamma(h+n)} \int D^{2h} X_0 \frac{X_0^{m_1} \dots X_0^{m_n}}{X_{10}^a X_{20}^b X_{03}^e X_{04}^f}, \quad (2.62)$$

where now $a+b+e+f=2h+n$. Combining denominators and applying (2.23), we obtain

$$I^{m_1 \dots m_n}(X_i) = \int_0^\infty \frac{d\beta}{\beta} \frac{d\gamma}{\gamma} \frac{d\delta}{\delta} \beta^b \gamma^e \delta^f \frac{X_{\beta,\gamma,\delta}^{m_1} \dots X_{\beta,\gamma,\delta}^{m_n}}{(-X_{\beta,\gamma,\delta}^2)^{n+h}} - \text{traces} \quad (2.63)$$

$$X_{\beta,\gamma,\delta} \equiv X_1 + \beta X_2 + \gamma X_3 + \delta X_4. \quad (2.64)$$

Finally, expanding the numerator in monomials, we can evaluate the result in terms of the scalar integrals (2.47),

$$\begin{aligned}
I^{m_1 \dots m_n}(X_i) &= \sum_{p+q+r+s=n} \frac{n!}{p!q!r!s!} I_{b+q,e+r,f+s}^{(h+n)}(X_i) \\
&\quad \times X_1^{(m_1)} \dots X_1^{m_p} X_2^{m_{p+1}} \dots X_2^{m_{p+q}} X_3^{m_{p+q+1}} \dots X_3^{m_{n-s}} X_4^{m_{n-s+1}} \dots X_4^{m_n} \\
&\quad - \text{traces.}
\end{aligned} \tag{2.65}$$

Since the X_i prefactors have trivial monodromy, projection onto the monodromy-invariant subspace can be performed termwise on each scalar integral $I_{b,e,f}^{(h)}$.

2.4 Higher Spin Conformal Blocks

2.4.1 General Method

With the language of section 2.2 and the results of section 2.3, computing higher-spin conformal blocks is a simple generalization of the case for scalar blocks. Consider a four-point function of primary operators ϕ_i in different Lorentz representations. The first step is to lift the operators ϕ_i to embedding space fields $\phi_i^{I_i}(X_i)$, where I_i is a general embedding space Lorentz index. The precise way to lift ϕ_i depends on its Lorentz representation and the spacetime dimension. We will give several concrete examples below.

Three-point functions of ϕ_i 's with an operator $\mathcal{O}^J(X)$ and its conjugate $\overline{\mathcal{O}}_J(X)$ are generically a sum of several tensor structures, each with its own independent OPE coefficient,

$$\langle \mathcal{O}^J \phi_1^{I_1} \phi_2^{I_2} \rangle = \langle \mathcal{O}^J \phi_1^{I_1} \phi_2^{I_2} \rangle^{(m)} \lambda_m, \tag{2.66}$$

$$\langle \overline{\mathcal{O}}_J \phi_3^{I_3} \phi_4^{I_4} \rangle = \langle \overline{\mathcal{O}}_J \phi_3^{I_3} \phi_4^{I_4} \rangle^{(n)} \eta_n. \tag{2.67}$$

Here, we have denoted the independent structures by a superscript $\langle \dots \rangle^{(m)}$, the associated OPE coefficients by λ_m and η_n , and a sum over m, n is implied. The number of structures in each three-point function depends on the Lorentz representations of \mathcal{O} and ϕ_i .

The four-point function $\langle \phi_1 \phi_2 \phi_3 \phi_4 \rangle$ has a conformal partial wave expansion

$$\langle \phi_1^{I_1}(X_1) \phi_2^{I_2}(X_2) \phi_3^{I_3}(X_3) \phi_4^{I_4}(X_4) \rangle = \sum_{\mathcal{O} \in \phi_1 \times \phi_2} \lambda_m \eta_n W_{\mathcal{O}}^{(m,n)I_1 I_2 I_3 I_4}(X_i), \quad (2.68)$$

where $W_{\mathcal{O}}^{(m,n)}$ is the conformal partial wave corresponding to the pair of tensor structures (m, n) . To compute $W_{\mathcal{O}}^{(m,n)}$, we need a conformally invariant projector $|\mathcal{O}|$ analogous to (2.35) which enables us to “sew together” the three-point functions (2.66) and (2.67).

In the cases we will encounter below, the embedding space lift of \mathcal{O}^J will have gauge-redundancies, which $|\mathcal{O}|$ must respect. Our projector will have the general form

$$|\mathcal{O}| = \int D^d X D^d Y |\mathcal{O}^J(X)\rangle \frac{\Pi(X, Y)_J^K}{(-2X \cdot Y)^{d+\deg \mathcal{O} + \deg \Pi}} \langle \overline{\mathcal{O}}_K(Y) |, \quad (2.69)$$

where $\Pi(X, Y)$ is a tensor built from X, Y that ensures gauge-invariance, and the denominator is chosen so that the projective integral is well-defined. Specifically, $\deg \mathcal{O}$ is the degree of the embedding-space lift of \mathcal{O} , and $\deg \Pi$ is the degree of $\Pi(X, Y)$ in either X or Y (which must be the same). After performing the integral, we must additionally project out the shadow contribution. The integrals we encounter in practice will always be reducible to a sum of basic tensor four-point integrals (2.62) whose monodromy projections we can evaluate with (2.65) and (2.55).

Inserting $|\mathcal{O}|$ within a four-point function is guaranteed by conformal invariance to produce a linear combination of conformal partial waves for the exchange of \mathcal{O} . To normalize them correctly, we should insert $|\mathcal{O}|$ within a three-point function, as in (2.39). In general, the projector can mix different tensor structures,

$$\langle \overline{\mathcal{O}}_J | \mathcal{O} | \phi_3^{I_3} \phi_4^{I_4} \rangle = \langle \overline{\mathcal{O}}_J | \phi_3^{I_3} \phi_4^{I_4} \rangle^{(m)} (\mathcal{M}_{\mathcal{O}34})_m^n \eta_n. \quad (2.70)$$

Thus, to obtain the conformal partial waves corresponding to a specific pair of tensor structures, we should multiply by the inverse of the mixing matrix $\mathcal{M}_{\mathcal{O}34}$,

$$W_{\mathcal{O}}^{m,n} = {}^{(m)} \langle \phi_1 \phi_2 | \mathcal{O} | \phi_3 \phi_4 \rangle^{(k)} (\mathcal{M}_{\mathcal{O}34}^{-1})_k^n. \quad (2.71)$$

With this prescription, $W_{\mathcal{O}}^{m,n}$ has the correct limiting behavior as $x_1 \rightarrow x_2$ (and $\phi_1\phi_2$ becomes better approximated by linear combinations of $\overline{\mathcal{O}}$). Since conformal partial waves are determined by either of the limits $x_1 \rightarrow x_2$ or $x_3 \rightarrow x_4$, the apparent asymmetry of (2.71) under $1, 2 \leftrightarrow 3, 4$ is illusory. Indeed, we must also have

$$W_{\mathcal{O}}^{m,n} = (\mathcal{M}_{\overline{\mathcal{O}}12}^{-1})_k{}^{m(k)} \langle \phi_1\phi_2 | \mathcal{O} | \phi_3\phi_4 \rangle^{(n)}. \quad (2.72)$$

In the examples below, the mixing matrix will be an overall constant, so the equivalence between (2.71) and (2.72) will be obvious.¹¹

2.4.2 Tensor Operators in the Embedding Space

The simplest operators to which we can apply this machinery are tensors. In this section, we focus on traceless tensors $\phi^{\mu_1 \dots \mu_\ell}$ whose Lorentz representations are specified by some pattern of symmetries in their indices. This is sufficient for understanding all bosonic operators in 3D CFTs, since these can always be decomposed into traceless symmetric representations of the Lorentz group. In higher than three dimensions, such tensors could be reducible (for instance, an antisymmetric tensor in four dimensions can be decomposed into anti-/self-dual parts), and it is convenient to use more refined techniques. We will develop them for 4d CFTs in section 2.5.

As argued in the previous section, two elements are required to compute conformal blocks for tensor operators: 1) a way to lift tensors to the embedding space, and 2) a gauge- and conformally-invariant projector. Embedding space lifts for tensors were introduced in [72] and further developed in [69, 73]. A primary operator $\phi^{\mu_1 \dots \mu_\ell}(x)$ with dimension Δ transforming as a traceless tensor of the Lorentz group can be lifted to an embedding space tensor $\Phi^{m_1 \dots m_\ell}(X)$ with the following properties:

¹¹It may be possible to show that this is always true, analogous to the arguments for scalar \mathcal{O} given in section 2.2.6. It would follow if $|\mathcal{O}|$ can be interpreted as a projection operator on a fixed Hilbert space in radial quantization. The fact that the monodromy M depends on the positions X_1, X_2 makes such an interpretation difficult.

1. defined on the null-cone,
2. traceless and possessing the same index symmetries as $\phi^{\mu_1 \dots \mu_\ell}$,
3. defined modulo tensors of the form $X^{m_i} \Lambda^{m_1 \dots \widehat{m_i} \dots m_\ell}(X)$,
4. transverse $X_{m_i} \Phi^{m_1 \dots m_i \dots m_\ell}(X) = 0$,
5. degree $-\Delta$ in X .

One can recover the original tensor $\phi^{\mu_1 \dots \mu_\ell}(x)$ by restricting to the Poincaré section $X^m = (1, x^2, x^\mu)$, and projecting indices as follows

$$\phi_{\mu_1 \dots \mu_\ell}(x) = \frac{\partial X^{m_1}}{\partial x^{\mu_1}} \dots \frac{\partial X^{m_\ell}}{\partial x^{\mu_\ell}} \Phi_{m_1 \dots m_\ell}(X). \quad (2.73)$$

When $\Phi^{m_1 \dots m_\ell}$ is symmetric in its indices (so that it transforms in a spin- ℓ representation of the Lorentz group), it is often convenient to use index-free fields

$$\Phi(X, Z) \equiv \Phi^{m_1 \dots m_\ell}(X) Z_{m_1} \dots Z_{m_\ell}, \quad (2.74)$$

which are homogeneous polynomials of degree ℓ in an auxiliary vector Z . Each property of $\Phi^{m_1 \dots m_\ell}$ is reflected in properties of $\Phi(X, Z)$. The tracelessness condition (2) means that we can restrict $\Phi(X, Z)$ to the null-cone $Z^2 = 0$ without losing any information. The redundancy (3) means that we can further restrict $\Phi(X, Z)$ to the plane $Z \cdot X = 0$. Finally, transverseness (4) implies that $\Phi(X, Z)$ has a gauge-redundancy under $Z \rightarrow Z + \lambda X$, for $\lambda \in \mathbb{R}$.

Correlators of symmetric tensors $\Phi_i(X_i, Z_i)$ must be gauge- and conformally-invariant functions of X_i and Z_i with the correct homogeneity properties. In two- and three-point correlators, such functions can be constructed as polynomials in the basic invariants

$$V_{i,jk} \equiv \frac{X_j \cdot Z_i}{X_{ij}} - \frac{X_k \cdot Z_i}{X_{ik}}, \quad H_{ij} \equiv \frac{(X_i \cdot X_j)(Z_i \cdot Z_j) - (X_i \cdot Z_j)(X_j \cdot Z_i)}{X_i \cdot X_j}, \quad (2.75)$$

along with the X_{ij} . In 3d and 4d, other invariants involving ϵ -tensors are possible [69]. They will not appear in the examples below.

2.4.3 Projectors for Tensor Operators

Given a tensor operator $\mathcal{O}^{m_1 \dots m_\ell}$, there is an essentially unique projector $|\mathcal{O}|$ of the form (2.69) compatible with all of the above properties,

$$|\mathcal{O}| = \int D^d X D^d Y |\mathcal{O}_{m_1 \dots m_\ell}(X)\rangle \frac{\prod_i (\eta^{m_i n_i}(X \cdot Y) - Y^{m_i} X^{n_i})}{(-2X \cdot Y)^{d-\Delta+\ell}} \langle \mathcal{O}_{n_1 \dots n_\ell}(Y) |. \quad (2.76)$$

The tensors $\eta^{m_i n_i}(X \cdot Y) - Y^{m_i} X^{n_i}$ are required to ensure invariance under gauge transformations $\mathcal{O}_{n_i \dots}(X) \rightarrow \mathcal{O}_{n_i \dots}(X) + X_{n_i} \Lambda \dots$.

To render (2.76) in a form more similar to (2.35), we can define the shadow operator

$$\tilde{\mathcal{O}}(X)^{m_1 \dots m_\ell} \equiv \int D^d Y \frac{\prod_i (\eta^{m_i n_i}(X \cdot Y) - Y^{m_i} X^{n_i})}{(-2X \cdot Y)^{d-\Delta+\ell}} \mathcal{O}_{n_1 \dots n_\ell}(Y). \quad (2.77)$$

In terms of $\tilde{\mathcal{O}}$, the projector $|\mathcal{O}|$ becomes simply

$$|\mathcal{O}| = \int D^d X |\mathcal{O}_{m_1 \dots m_\ell}(X)\rangle \langle \tilde{\mathcal{O}}^{m_1 \dots m_\ell}(X) |. \quad (2.78)$$

Using index-free notation for symmetric tensors, the shadow operator can be written

$$\tilde{\mathcal{O}}(X, Z) = \int D^d Y \frac{1}{(-2X \cdot Y)^{d-\Delta+\ell}} \mathcal{O}(Y, C_{ZX} \cdot Y), \quad (2.79)$$

where $C_{ZX}^{mn} \equiv Z^m X^n - X^m Z^n$. Note that $\tilde{\mathcal{O}}$ is well-defined, since $Y \cdot C_{ZX} \cdot Y = 0$ and $(C_{ZX} \cdot Y)^2 = 0$ (assuming that $Z \cdot X = 0$). Further, $\tilde{\mathcal{O}}(X, Z)$ automatically enjoys the correct gauge redundancy, since C_{ZX} is invariant under $Z \rightarrow Z + \lambda X$ s. Finally, since $\tilde{\mathcal{O}}(X, Z)$ has degrees $-(d - \Delta)$ and ℓ in X and Z , it formally possesses all the required properties of a primary operator with dimension $d - \Delta$ and spin ℓ .

Before moving on to examples, let us quickly summarize the approach of [69] for computing conformal blocks of symmetric tensors. The authors define differential operators $\mathcal{D}_{\text{left}}^{(m)}$ and $\mathcal{D}_{\text{right}}^{(n)}$ that turn three-point functions of scalars φ_i into three-point functions of higher-spin operators ϕ_i ,

$$^{(m)}\langle \phi_1(X_1, Z_1) \phi_2(X_2, Z_2) \mathcal{O}(X, Z) \rangle = \mathcal{D}_{\text{left}}^{(m)} \langle \varphi_1(X_1) \varphi_2(X_2) \mathcal{O}(X, Z) \rangle \quad (2.80)$$

$$^{(n)}\langle \phi_3(X_3, Z_3) \phi_4(X_4, Z_4) \mathcal{O}(X, Z) \rangle = \mathcal{D}_{\text{right}}^{(n)} \langle \varphi_3(X_3) \varphi_4(X_4) \mathcal{O}(X, Z) \rangle. \quad (2.81)$$

Here m, n index the possible tensor structures. $\mathcal{D}_{\text{left}}^{(m)}$ and $\mathcal{D}_{\text{right}}^{(n)}$ are constructed to involve only the external coordinates X_i, Z_i . By linearity, and the fact that the \mathcal{D} 's act trivially under monodromy, it's clear that

$$\begin{aligned} & {}^{(m)}\langle \phi_1(X_1, Z_1) \phi_2(X_2, Z_2) | \mathcal{O} | \phi_3(X_3, Z_3) \phi_4(X_4, Z_4) \rangle^{(n)} \\ &= \mathcal{D}_{\text{left}}^{(m)} \mathcal{D}_{\text{right}}^{(n)} \langle \varphi_1(X_1) \varphi_2(X_2) | \mathcal{O} | \varphi_3(X_3) \varphi_4(X_4) \rangle, \end{aligned} \quad (2.82)$$

so conformal partial waves for \mathcal{O} exchanged between $\phi_i(X_i, Z_i)$ are derivatives of conformal partial waves for \mathcal{O} exchanged between scalars $\varphi_i(X_i)$. A virtue of this approach is that expressions for lower-spin blocks can be reused in computations of higher-spin blocks. However, external derivatives cannot change the conformal multiplet of the operator \mathcal{O} being exchanged. One must always begin with a “seed” calculation of some nonzero conformal block involving a given \mathcal{O} .

2.4.4 Example: Antisymmetric Tensor Exchange

The simplest tensor conformal block that is not related via derivatives to a scalar block is the exchange of an antisymmetric tensor F^{mn} in a four-point function of two scalars and two vectors $\langle \phi_1 J_2^l \phi_3 J_4^k \rangle$. (In a four-point function with fewer than two vectors, any pairing of the operators would include a pair of scalars. The OPE of two scalars contains only symmetric tensors, so only these would contribute in the conformal block expansion.) We work in d dimensions and assume that F^{mn} transforms irreducibly under the Lorentz group. (Although this is incorrect when $d = 4$, the result still applies if the self-dual and anti-self-dual parts of F have the same OPE coefficient in $\phi_1 \times J_2$ and $\phi_3 \times J_4$.)

The three-point function $\langle F^{mn} \phi_3 J_4^l \rangle$ has a unique allowed tensor structure

$$\langle F^{mn}(X_0) \phi_3(X_3) J_4(X_4, Z_4) \rangle = \frac{((X_0 \cdot X_4) Z_4^m - (X_0 \cdot Z_4) X_4^m)(X_{03} X_4^n - X_{04} X_3^n) - (m \leftrightarrow n)}{X_{03}^{\frac{\Delta+\Delta_3-\Delta_4+1}{2}} X_{04}^{\frac{\Delta+\Delta_4-\Delta_3+3}{2}} X_{34}^{\frac{\Delta_3+\Delta_4-\Delta+1}{2}}}, \quad (2.83)$$

where Δ is the dimension of F , and we are using index-free notation (2.74) for J_4 . This expression is fixed by homogeneity and the requirement of transverseness in its indices, up to gauge redundant terms proportional to X_0^m, X_0^n , which we have dropped.

From the definition (2.77), we can compute the shadow transform

$$\langle \tilde{F}^{mn}(X_0)\phi_3(X_3)J_4(X_4, Z_4) \rangle = \mathcal{S}_\Delta \langle F^{mn}(X_0)\phi_3(X_3)J_4(X_4, Z_4) \rangle|_{\Delta \rightarrow \tilde{\Delta}}, \quad (2.84)$$

$$\text{where } \mathcal{S}_\Delta \equiv \frac{\pi^h(\Delta-2)\Gamma(\Delta-h)}{4\Gamma(\tilde{\Delta}+1)} \frac{\Gamma(\frac{\tilde{\Delta}+\Delta_{34}+1}{2})\Gamma(\frac{\tilde{\Delta}-\Delta_{34}+1}{2})}{\Gamma(\frac{\Delta+\Delta_{34}+1}{2})\Gamma(\frac{\Delta-\Delta_{34}+1}{2})} \quad (2.85)$$

and $h = d/2$, $\tilde{\Delta} = d - \Delta$ as usual. As expected, the result has the correct form for a three-point function of ϕ_3 and J_4 with an antisymmetric tensor of dimension $\tilde{\Delta}$.

The next step is to determine the mixing matrix $(\mathcal{M}_{F34})_i^j$ from inserting the projector $|F|$ within a three-point function. Since there is only a single allowed three-point structure, this is just a number \mathcal{M} . We can compute it with a simple trick. Suppose F^{mn} is normalized to have two-point function

$$\langle F^{mn}(X_1)F^{kl}(X_2) \rangle = \frac{1}{2} \frac{((X_1 \cdot X_2)\eta^{mk} - X_2^m X_1^k)((X_1 \cdot X_2)\eta^{nl} - X_2^n X_1^l) - (m \leftrightarrow n)}{X_{12}^{\Delta+2}}. \quad (2.86)$$

Again, this structure is fixed up to gauge redundancy by homogeneity and transverseness. Note that the numerator has precisely the same form as the tensor appearing in the definition of $|F|$ (2.76). In fact, inserting $|F|$ between the two-point function (2.86) and three-point function (2.83) is equivalent to simply iterating the shadow transform twice. We can read off the result from (2.84),

$$\begin{aligned} \mathcal{M} \langle F\phi_3 J_4 \rangle &= \langle F|F|\phi_3 J_4 \rangle \\ &= \langle \tilde{\tilde{F}}\phi_3 J_4 \rangle \\ &= \mathcal{S}_\Delta \mathcal{S}_{\tilde{\Delta}} \langle F\phi_3 J_4 \rangle \\ &= \frac{\pi^{2h}(\tilde{\Delta}-2)(\Delta-2)\Gamma(\tilde{\Delta}-h)\Gamma(\Delta-h)}{16\Gamma(\tilde{\Delta}+1)\Gamma(\Delta+1)} \langle F\phi_3 J_4 \rangle. \end{aligned} \quad (2.87)$$

Applying (2.71), the conformal partial wave corresponding to exchange of F^{mn} is given by

$$\begin{aligned} & \left(\frac{X_{14}}{X_{13}} \right)^{\frac{\Delta_{34}}{2}} \left(\frac{X_{24}}{X_{14}} \right)^{\frac{\Delta_{12}}{2}} \frac{g_F^{\Delta_i}(u, v)}{X_{12}^{\frac{\Delta_1 + \Delta_2}{2}} X_{34}^{\frac{\Delta_3 + \Delta_4}{2}}} \\ &= \frac{1}{\mathcal{M}} \langle \phi_1(X_1) J_2(X_2, Z_2) | F | \phi_3(X_3) J_4(X_4, Z_4) \rangle \end{aligned} \quad (2.88)$$

$$= \frac{1}{\mathcal{S}_{\tilde{\Delta}}} \int D^d X \langle \phi_1(X_1) J_2(X_2, Z_2) F^{mn}(X) \rangle \left(\langle F_{mn}(X) \phi_3(X_3) J_4(X_4, Z_4) \rangle |_{\Delta \rightarrow \tilde{\Delta}} \right). \quad (2.89)$$

Finally, using (2.65) to evaluate the integral, we obtain

$$\begin{aligned} g_F^{\Delta_i}(u, v) &= \frac{2u^{\Delta/2-1/2} X_{24} \Gamma(\Delta + 1)}{(2 - \tilde{\Delta}) \Gamma(\alpha + 1) \Gamma(\beta + 1) \Gamma(\Delta - \alpha) \Gamma(\Delta - \beta) \Gamma(h - \Delta)} \\ &\times \left[V_{2,14} V_{4,12} \left((v-1)(v J_{0,1,0}^{(2)} + v J_{1,1,1}^{(2)} - J_{0,0,1}^{(2)}) - \alpha J_{0,1,1}^{(1)} - \alpha v J_{0,1,0}^{(1)} \right. \right. \\ &\quad + (\beta - \Delta + h)(\alpha J_{1,1,1}^{(0)} - J_{1,0,1}^{(1)}) - v(\alpha - \Delta + 1)(J_{2,1,1}^{(1)} + J_{1,1,0}^{(1)}) \\ &\quad \left. \left. - (\beta - h + 1)v(J_{1,2,1}^{(1)} + (\alpha - \Delta + 1)v J_{2,2,1}^{(0)}) - 2v(\alpha + \beta - \Delta + 1)J_{1,1,1}^{(1)} \right) \right. \\ &\quad + V_{2,14} V_{4,23} (v-1) \left(J_{0,0,0}^{(2)} + v J_{0,1,0}^{(2)} - \alpha J_{0,1,0}^{(1)} \right) \\ &\quad + V_{2,34} V_{4,12} \left(-\frac{v-1}{h+1} (J_{0,0,0}^{(2)} + J_{0,0,1}^{(2)} + J_{0,1,1}^{(2)} + J_{1,0,1}^{(2)} + v J_{1,1,1}^{(2)} + u J_{1,1,2}^{(2)}) \right. \\ &\quad + (\beta - \Delta + h)(\alpha J_{1,1,1}^{(0)} + (\alpha - \Delta + 1)J_{2,1,1}^{(0)}) \\ &\quad \left. + (\beta - h + 1)(\alpha J_{1,2,1}^{(0)} + (\alpha - \Delta + 1)v J_{2,2,1}^{(0)}) \right) \\ &\quad + V_{2,34} V_{4,23} v \left(-(v-1)(J_{0,1,0}^{(2)} + J_{1,1,1}^{(2)}) + (\beta - h + 1)(J_{1,2,1}^{(1)} + (\alpha - \Delta + 1)J_{2,2,1}^{(0)}) \right. \\ &\quad \left. + \alpha J_{0,1,0}^{(1)} + (\alpha + \beta - \Delta + h)J_{1,1,1}^{(1)} + (\alpha - \Delta + 1)(J_{1,1,0}^{(1)} + J_{2,1,1}^{(1)}) \right) \\ &\quad \left. + \frac{v H_{24}}{X_{24}} \left(-(v-1)J_{0,1,0}^{(2)} + \alpha J_{0,1,0}^{(1)} + (\alpha - \Delta + 1)J_{1,1,0}^{(1)} \right) \right], \end{aligned} \quad (2.90)$$

where

$$\alpha \equiv \frac{\Delta - \Delta_{12} - 1}{2}, \quad \beta \equiv \frac{\Delta + \Delta_{34} - 1}{2}, \quad (2.91)$$

$V_{i,jk}, H_{ij}$ are given by (2.75), and the $J_{j,k,l}^{(i)}$ are shorthand for monodromy-projected conformal

four-point integrals,

$$\begin{aligned}
J_{j,k,l}^{(i)} &\equiv \Gamma(h+i)(X_{14}^{b+e-h-i} X_{13}^{f-h-i} X_{34}^{h+i-f-e} X_{24}^{-b})^{-1} I_{b,e,f}^{(h+i)}|_{M=1}, \text{ with} \\
b &= \alpha + i + j - 1 \\
e &= \beta + h + i + k - l \\
f &= 1 - \beta + h - k.
\end{aligned} \tag{2.92}$$

The powers of X_{ij} in the definition of J have been chosen so that J is a function of conformal cross-ratios. In general dimensions, it is given by the expression (2.55); when d is even, it can be written in terms of products of hypergeometric functions using (2.58).

2.5 Twistor Methods for 4d CFTs

2.5.1 Lifting Spinors to the Embedding Space

Although the methods of the previous section are sufficient for computations involving tensors, we need a more flexible formalism to deal with more general Lorentz representations. For the remainder of this work, we focus on CFTs in four dimensions, where twistors provide natural building blocks for conformal invariants.¹²

Twistor space $\mathbb{T} \cong \mathbb{C}^4$ consists of four-component objects

$$Z_A = \begin{pmatrix} \lambda_\alpha \\ \mu^{\dot{\alpha}} \end{pmatrix} \tag{2.93}$$

transforming as left-chiral spinors of the conformal group $\text{SO}(4,2)$, or equivalently fundamentals of $\text{SU}(2,2)$. \mathbb{T} possesses a totally antisymmetric conformal invariant given by the determinant

¹²In this and subsequent sections, we work with 4d spinors in signature $-+++$. Conformal integrals can be defined by analytic continuation back to Euclidean signature. Our conventions for spinors and Γ -matrices in the embedding space are detailed in Appendix 2.B.

$\langle Z_1 Z_2 Z_3 Z_4 \rangle \equiv \epsilon^{ABCD} Z_{1A} Z_{2B} Z_{3C} Z_{4D}$. We also have the dual space $\overline{\mathbb{T}}$ with coordinates \overline{W}^A , and an invariant pairing $\overline{W}Z = \overline{W}^A Z_A$.

The (complexified) embedding space itself is the antisymmetric tensor-square of twistor space, $\mathbb{C}^6 \cong \wedge^2 \mathbb{T}^4$, and the null-cone consists of precisely the pure tensors (or “simple bitwistors”) under this identification,

$$X_{AB} = Z_A W_B - Z_B W_A, \quad (2.94)$$

where $X_{AB} \equiv X_m \Gamma_{AB}^m$, with Γ_{AB}^m a chiral gamma-matrix. In other words, the projective null-cone is isomorphic to the Grassmanian of two-planes in twistor space $\text{Gr}(2, \mathbb{T})$. Note that the null condition $X^2 = 0$ implies that $X\overline{X} = \overline{X}X = 0$, where $\overline{X}^{AB} \equiv X_m \overline{\Gamma}^{mAB} = \frac{1}{2} \epsilon^{ABCD} X_{CD}$.

Arbitrary 4d Lorentz representations can be built from products of spinors. So if we can lift spinor operators to the embedding space, we can lift any representation. As shown in [73], spinors lift to twistors. Specifically, given a spinor primary $\psi_\alpha(x)$ with dimension Δ , the combination

$$\Psi_A(X) \equiv (X^+)^{1/2-\Delta} \begin{pmatrix} \psi_\alpha(x) \\ i(x \cdot \overline{\sigma})^{\dot{\alpha}\beta} \psi_\beta(x) \end{pmatrix}, \quad (2.95)$$

with $x^\mu = X^\mu/X^+$, transforms as a twistor under the conformal group. By construction, $\Psi_A(X)$ satisfies the transverseness condition $\overline{X}^{AB} \Psi_B(X) = 0$, and has degree $1/2 - \Delta$ in X .

It is convenient to use a slightly different (but equivalent) lift of $\psi_\alpha(x)$. Note that we can always solve the transverseness condition $\overline{X}\Psi = 0$ as $\Psi = X\overline{\Psi}$ for some $\overline{\Psi} \in \overline{\mathbb{T}}$. In turn, $\overline{\Psi}$ is defined modulo twistors of the form $\overline{X}Z$, $Z \in \mathbb{T}$. This follows because the multiplication maps $X : \overline{\mathbb{T}} \rightarrow \mathbb{T}$ and $\overline{X} : \mathbb{T} \rightarrow \overline{\mathbb{T}}$ have rank two and compose to zero, so that

$$\ker(\overline{X}) = \text{im}(X) \cong \overline{\mathbb{T}} / \ker(X) = \overline{\mathbb{T}} / \text{im}(\overline{X}). \quad (2.96)$$

Solving the transverseness equation for (2.95) in this way, we lift $\psi_\alpha(x)$ to a gauge-redundant

dual-twistor of degree $-1/2 - \Delta$,

$$\psi_\alpha(x) \rightarrow \overline{\Psi}^A(X), \quad \text{where} \quad \overline{\Psi}(X) \sim \overline{\Psi}(X) + \overline{X}Z. \quad (2.97)$$

Similarly, right-chiral spinors $\overline{\lambda}_{\dot{\alpha}}(x)$ lift to twistors $\Lambda_A(X)$ of degree $-1/2 - \Delta$ with a gauge-redundancy $\Lambda(X) \sim \Lambda(X) + X\overline{Z}$.

The relation between the original four-dimensional fields and their twistor counterparts is extremely simple,

$$\psi_\alpha(x) = X_{\alpha B} \overline{\Psi}^B(X)|_{X=(1, x^2, x^\mu)} \quad (2.98)$$

$$\overline{\lambda}_{\dot{\alpha}}(x) = \overline{X}_{\dot{\alpha}}{}^B \Lambda_B(X)|_{X=(1, x^2, x^\mu)}, \quad (2.99)$$

where in each case we restrict X to the Poincaré section. As an example, a two-point function of twistor fields is fixed by conformal invariance and homogeneity to have the form

$$\langle \overline{\Psi}^A(X) \Lambda_B(Y) \rangle = \frac{\delta_B^A}{(-2X \cdot Y)^{\Delta+1/2}}, \quad (2.100)$$

where the gauge-redundancies of $\overline{\Psi}$ and Λ let us discard terms proportional to $\overline{X}^{AC} Y_{CB} = -2X \cdot Y - \overline{Y}^{AC} X_{CB}$. Applying the dictionary (2.98, 2.99), we find

$$\langle \psi_\alpha(x) \overline{\lambda}_{\dot{\beta}}(y) \rangle = -\frac{X_{\alpha A} \overline{Y}^A_{\dot{\beta}}}{(-2X \cdot Y)^{\Delta+1/2}} = -\frac{i(x-y)_{\alpha\dot{\beta}}}{(x-y)^{2\Delta+1}}, \quad (2.101)$$

which is precisely the correct form for a two-point function of spinor primaries.

More general operators $\mathcal{O}_{\alpha_1 \dots \alpha_j}^{\dot{\beta}_1 \dots \dot{\beta}_{\overline{j}}}(x)$ in $(j/2, \overline{j}/2)$ representations of the Lorentz group lift to symmetric multi-twistors $\mathcal{O}_{B_1 \dots B_{\overline{j}}}^{A_1 \dots A_j}(X)$ of degree $-\Delta - j/2 - \overline{j}/2$ in X , subject to a gauge-redundancy in each index. As in section 2.4.2, it will often be useful to adopt index-free notation

$$\mathcal{O}(X, S, \overline{S}) \equiv \mathcal{O}_{B_1 \dots B_{\overline{j}}}^{A_1 \dots A_j}(X) S_{A_1} \dots S_{A_j} \overline{S}^{B_1} \dots \overline{S}^{B_{\overline{j}}} \quad (2.102)$$

where S and \overline{S} are auxiliary twistors. In this language, the gauge-redundancy of \mathcal{O} means that we can restrict S, \overline{S} to be transverse

$$\overline{X}S = 0, \quad X\overline{S} = 0. \quad (2.103)$$

Of course, these conditions can always be solved as $S = X\bar{T}$, $\bar{S} = \bar{X}T$ for some T, \bar{T} . Consequently, the product $\bar{S}S$ vanishes as well. Going back to explicit indices, this means that $\mathcal{O}_{B_1\dots B_{\bar{j}}}^{A_1\dots A_j}$ must also have a gauge redundancy under shifts proportional to $\delta_{B_{\bar{i}}}^{A_i}$.

Given a multi-twistor field $\mathcal{O}(X, S, \bar{S})$, one can project back to four-dimensions as follows,

$$\mathcal{O}_{\alpha_1\dots\alpha_j}^{\dot{\alpha}_1\dots\dot{\alpha}_{\bar{j}}}(x) \equiv \frac{1}{j!\bar{j}!} \left(X \frac{\partial}{\partial S}\right)_{\alpha_1} \cdots \left(X \frac{\partial}{\partial S}\right)_{\alpha_j} \left(\bar{X} \frac{\partial}{\partial \bar{S}}\right)^{\dot{\alpha}_1} \cdots \left(\bar{X} \frac{\partial}{\partial \bar{S}}\right)^{\dot{\alpha}_{\bar{j}}} \mathcal{O}(X, S, \bar{S}), \quad (2.104)$$

where we restrict X to the Poincaré section.

As a special case, a vector operator $j^\mu(x)$ can be represented in the embedding space either as a multi-twistor $J_A^B(X)$, or as a vector $\mathcal{J}_m(X)$ satisfying the conditions of section 2.4.2. The relation between these two formalisms is

$$\bar{X}^{AC} J_C^B(X) - \bar{X}^{BC} J_C^A(X) = -i\bar{\Gamma}^{mAB} \mathcal{J}_m. \quad (2.105)$$

This is consistent with the gauge redundancies in both descriptions. The transformation $J_A^B \rightarrow J_A^B + X_{AC}\Lambda^{CB}$ acts trivially on the left-hand side, while the redundancies $J_A^B \rightarrow J_A^B + \lambda\delta_A^B$ and $J_A^B \rightarrow J_A^B + \bar{X}^{BC}\Lambda_{CA}$ become shifts $\mathcal{J}_m \rightarrow \mathcal{J}_m + X_m$.

2.5.2 Two-Point and Three-Point Functions

In this section, we identify the basic ingredients for two- and three-point correlators of multi-twistor operators $\mathcal{O}(X, S, \bar{S})$. Given the condition (2.103), only one type of conformal invariant other than X_{ij} can appear in a two-point function,

$$I_{i\bar{j}} \equiv S_i \bar{S}_{\bar{j}}, \quad i \neq j.$$

For example, as we saw above, a two-point function of spinors is given by

$$\langle \Psi(X_1, S_1) \bar{\Psi}(X_2, \bar{S}_2) \rangle = \frac{I_{1\bar{2}}}{X_{12}^{\Delta+1/2}}. \quad (2.106)$$

Similarly, a dimension- Δ operator $\mathcal{O}(X, S, \bar{S})$ transforming in a $(j/2, \bar{j}/2)$ representation of the Lorentz group has two-point function

$$\langle \mathcal{O}(X_1, S_1, \bar{S}_1) \bar{\mathcal{O}}(X_2, S_2, \bar{S}_2) \rangle = \frac{I_{12}^j I_{21}^{\bar{j}}}{X_{12}^{\Delta+j/2+\bar{j}/2}}, \quad (2.107)$$

where $\bar{\mathcal{O}}$ transforms in the $(\bar{j}/2, j/2)$ Lorentz representation.

More invariants are possible in three-point correlators:

$$J_{i,jk} \equiv S_i \bar{X}_j X_k \bar{S}_i \quad (2.108)$$

$$K_{ijk} \equiv S_i \bar{X}_j S_k \quad (2.109)$$

$$\bar{K}_{ijk} \equiv \bar{S}_i X_j \bar{S}_k, \quad (2.110)$$

where each vanishes unless $i \neq j \neq k$. $J_{i,jk}$ is antisymmetric in its last two indices, while K_{ijk} and \bar{K}_{ijk} are antisymmetric under the exchange $i \leftrightarrow k$.

General three-point functions can be constructed from the invariants I, J, K, \bar{K} , along with the X_{ij} . However, these invariants are not algebraically independent. For instance, one can verify the relations

$$K_{123} \bar{K}_{231} = I_{3\bar{2}} J_{1,23} - X_{23} I_{3\bar{1}} I_{1\bar{2}} \quad (2.111)$$

$$J_{2,31} K_{123} = I_{1\bar{2}} K_{312} X_{23} - I_{3\bar{2}} K_{231} X_{12} \quad (2.112)$$

$$J_{2,31} \bar{K}_{123} = I_{2\bar{3}} \bar{K}_{231} X_{12} - I_{2\bar{1}} \bar{K}_{312} X_{23} \quad (2.113)$$

$$\begin{aligned} J_{1,23} J_{2,31} J_{3,12} = & X_{12} X_{23} X_{31} (I_{1\bar{2}} I_{2\bar{3}} I_{3\bar{1}} - I_{1\bar{3}} I_{2\bar{1}} X_{3\bar{2}}) \\ & - I_{1\bar{3}} I_{3\bar{1}} J_{2,31} X_{12} X_{23} - I_{2\bar{3}} I_{3\bar{2}} J_{1,23} X_{12} X_{31} - I_{1\bar{2}} I_{2\bar{1}} J_{3,12} X_{23} X_{31}, \end{aligned} \quad (2.114)$$

and arbitrary permutations of the labels $\{1, 2, 3\}$. Additional relations are possible (in addition to those generated by the above). We will not attempt to classify them here.¹³

¹³In verifying (2.111-2.114), it's extremely convenient to use twistor coordinates on the null-cone $X_{AB} = Z_A W_B -$

The general form of any correlator is determined by which combinations of gauge- and conformal-invariants have the correct homogeneity properties. As an example, let us consider a symmetric tensor $J(X, S, \bar{S})$ of spin- ℓ and dimension Δ and its correlators with scalars. The two-point function $\langle J(X_1, S_1, \bar{S}_1) J(X_2, S_2, \bar{S}_2) \rangle$ is given by (2.107) with $j = \bar{j} = \ell$. Projecting to flat space with (2.104), this becomes

$$\langle J_{\dot{\alpha}_1 \dots \dot{\alpha}_\ell}^{\alpha_1 \dots \alpha_\ell}(x) J_{\dot{\beta}_1 \dots \dot{\beta}_\ell}^{\beta_1 \dots \beta_\ell}(0) \rangle = \frac{x^{(\alpha_1}_{\dot{\beta}_1} \dots x^{\alpha_\ell)}_{\dot{\beta}_\ell} x^{(\beta_1}_{\dot{\alpha}_1} \dots x^{\beta_\ell)}_{\dot{\alpha}_\ell}}{x^{2(\Delta+\ell)}}. \quad (2.118)$$

This normalization differs from the one in [64, 65], $J_{\text{ours}} = (-2)^{-\ell/2} J_{\text{theirs}}$. As a consequence, our conformal block normalizations will differ as well.

The only structure with the correct homogeneity properties for a three-point function of J with scalars ϕ_1, ϕ_2 is

$$\langle \phi_1(X_1) \phi_2(X_2) J(X_3, S_3, \bar{S}_3) \rangle = \lambda \frac{J_{3,12}^\ell}{X_{12}^{\frac{\Delta_1+\Delta_2-\Delta+\ell}{2}} X_{23}^{\frac{\Delta+\Delta_2-\Delta_1+\ell}{2}} X_{13}^{\frac{\Delta+\Delta_1-\Delta_2+\ell}{2}}}, \quad (2.119)$$

where λ is an OPE coefficient. Restricting to the Poincaré section, and applying (2.104), this takes the familiar form

$$\langle \phi_1(x_1) \phi_2(x_2) J_{\dot{\alpha}_1 \dots \dot{\alpha}_\ell}^{\alpha_1 \dots \alpha_\ell}(x_3) \rangle = \lambda \frac{V_3^{(\alpha_1}_{\dot{\alpha}_1} \dots V_3^{\alpha_\ell)}_{\dot{\alpha}_\ell}}{x_{12}^{\frac{\Delta_1+\Delta_2-\Delta+\ell}{2}} x_{23}^{\frac{\Delta+\Delta_2-\Delta_1-\ell}{2}} x_{13}^{\frac{\Delta+\Delta_1-\Delta_2-\ell}{2}}}, \quad (2.120)$$

$$\text{where } V_{3\alpha\dot{\alpha}} = -\frac{(X_3 \bar{X}_1 X_2 \bar{X}_3)_{\alpha\dot{\alpha}}}{X_{13} X_{23}} = i \frac{(x_{31})_{\alpha\dot{\alpha}}}{x_{31}^2} - i \frac{(x_{32})_{\alpha\dot{\alpha}}}{x_{32}^2}. \quad (2.121)$$

With the normalization convention (2.118), J is imaginary when ℓ is odd, so that λ is always real in a unitary theory.

$Z_B W_A$. Auxiliary spinors S, \bar{S} can then be written

$$S_A = \alpha Z_A + \beta W_A \quad (2.115)$$

$$\bar{S}^A = \epsilon^{ABCD} Z_B W_C (\gamma T_D + \delta U_D) \quad (2.116)$$

for constants $\alpha, \beta, \gamma, \delta \in \mathbb{C}$. Here T, U are any two linearly-independent twistors, defined modulo Z, W . For instance in an n -point function, we are free to choose $T_i = Z_{i+1}, U_i = W_{i+1}$. One can additionally use the $\text{GL}(2, \mathbb{C})$ redundancy rotating Z into W to set β to zero. Relations between invariants then follow from the Schouten identity

$$0 = \langle 1234 \rangle \langle 5 | + \langle 2345 \rangle \langle 1 | + \langle 3451 \rangle \langle 2 | + \langle 4512 \rangle \langle 3 | + \langle 5123 \rangle \langle 4 |, \quad (2.117)$$

which expresses the fact that any five twistors are linearly dependent.

2.5.3 Twistor Projectors and Shadows

Given a multi-twistor operator \mathcal{O} with dimension Δ and Lorentz representation (j, \bar{j}) , there is an essentially unique gauge- and conformally-invariant projector,

$$|\mathcal{O}| = \frac{1}{j!2\bar{j}!2} \int D^4X D^4Y |\mathcal{O}(X, S, \bar{S})\rangle \frac{(\vec{\partial}_S X \vec{Y} \vec{\partial}_{\bar{T}})^j (\vec{\partial}_{\bar{S}} \bar{X} Y \vec{\partial}_T)^{\bar{j}}}{(-2X \cdot Y)^{4-\Delta+j/2+\bar{j}/2}} \langle \bar{\mathcal{O}}(Y, T, \bar{T})| \quad (2.122)$$

$$= \frac{1}{j!2\bar{j}!2} \int D^4X |\mathcal{O}(X, S, \bar{S})\rangle (\vec{\partial}_S X \vec{\partial}_T)^j (\vec{\partial}_{\bar{S}} \bar{X} \vec{\partial}_{\bar{T}})^{\bar{j}} \langle \tilde{\mathcal{O}}(X, T, \bar{T})|, \quad (2.123)$$

The products $X\bar{Y}$ and $\bar{X}Y$ in the numerator are required to project away gauge-dependent pieces. They are analogous to the factors $\eta^{mn}(X \cdot Y) - Y^m X^n$ in the tensor projector (2.76). In the second line, we have defined the the shadow operator $\tilde{\mathcal{O}}$ by

$$\tilde{\mathcal{O}}(X, S, \bar{S}) \equiv \int D^4Y \frac{1}{(-2X \cdot Y)^{4-\Delta+j/2+\bar{j}/2}} \bar{\mathcal{O}}(Y, Y\bar{S}, \bar{Y}S). \quad (2.124)$$

Note that $Y\bar{S}$ and $\bar{Y}S$ are automatically transverse with respect to Y , so that $\tilde{\mathcal{O}}$ is well-defined. Formally, $\tilde{\mathcal{O}}$ has the properties of a primary operator of dimension $4 - \Delta$ transforming in the $(\bar{j}/2, j/2)$ representation of the Lorentz group. As usual, this is useful in constraining the form of correlators involving $\tilde{\mathcal{O}}$.

2.5.4 Example: Spin- ℓ Exchange between Scalars

With the projector $|\mathcal{O}|$ in hand, we can specialize the procedure in section 2.4.1 to compute conformal blocks of multi-twistor operators. As a first example, let us reproduce the known conformal block for exchange of a spin- ℓ operator J with dimension Δ between scalars ϕ_i . We will assume that J is normalized as in (2.107).

Beginning with the three-point function

$$\langle J(X_0, S, \bar{S}) \phi_3(X_3) \phi_4(X_4) \rangle = \frac{(S\bar{X}_3 X_4 \bar{S})^\ell}{X_{34}^{\frac{\Delta_3+\Delta_4-\Delta+\ell}{2}} X_{03}^{\frac{\Delta+\Delta_{34}+\ell}{2}} X_{04}^{\frac{\Delta-\Delta_{34}+\ell}{2}}}, \quad (2.125)$$

one can compute

$$\begin{aligned} \langle \tilde{J}(X_0, T, \bar{T}) \phi_3(X_3) \phi_4(X_4) \rangle = & (-1)^\ell \frac{\pi^2 \Gamma(\Delta + \ell - 1)}{\Gamma(\tilde{\Delta} + \ell)(\Delta - 2)} \frac{\Gamma\left(\frac{\tilde{\Delta} - \Delta_{34} + \ell}{2}\right) \Gamma\left(\frac{\tilde{\Delta} + \Delta_{34} + \ell}{2}\right)}{\Gamma\left(\frac{\Delta - \Delta_{34} + \ell}{2}\right) \Gamma\left(\frac{\Delta + \Delta_{34} + \ell}{2}\right)} \\ & \times \frac{(T \bar{X}_3 X_4 \bar{T})^\ell}{X_{34}^{\frac{\Delta_3 + \Delta_4 - \tilde{\Delta} + \ell}{2}} X_{03}^{\frac{\tilde{\Delta} + \Delta_{34} + \ell}{2}} X_{04}^{\frac{\tilde{\Delta} - \Delta_{34} + \ell}{2}}}, \end{aligned} \quad (2.126)$$

where $\tilde{\Delta} = 4 - \Delta$. As expected, this has the form of a three-point function between scalars and a spin- ℓ operator of dimension $\tilde{\Delta}$.

Before sewing three-point correlators to compute a conformal block, we must determine the correct normalization factor. (Since there is a unique allowed three-point structure, the mixing matrix \mathcal{M} is simply an overall constant.) Inserting the projector $|J|$ between a two- and a three-point function is equivalent to iterating the shadow transform twice, so we can read off the correct normalization factor from (2.126),

$$\begin{aligned} \mathcal{M} \langle J \phi_3 \phi_4 \rangle &= \langle J | J | \phi_3 \phi_4 \rangle \\ &= \langle \tilde{J} \phi_3 \phi_4 \rangle \\ &= \frac{\pi^2 \Gamma(\Delta + \ell - 1)}{\Gamma(\tilde{\Delta} + \ell)(\Delta - 2)} \frac{\pi^2 \Gamma(\tilde{\Delta} + \ell - 1)}{\Gamma(\Delta + \ell)(2 - \Delta)} \langle J \phi_3 \phi_4 \rangle \end{aligned} \quad (2.127)$$

Finally, the numerator of our conformal block integral is given by

$$(S \bar{X}_1 X_2 \bar{S})^\ell \frac{(\vec{\partial}_S X_0 \vec{\partial}_T)^j (\vec{\partial}_{\bar{S}} \bar{X}_0 \vec{\partial}_{\bar{T}})^{\bar{j}}}{\ell!^4} (T \bar{X}_3 X_4 \bar{T})^\ell = (-1)^\ell s^{\ell/2} C_\ell^1(t), \quad (2.128)$$

where $C_\ell^\lambda(t)$ are Gegenbauer polynomials and

$$t \equiv \frac{-X_{13} X_{02} X_{04}}{2\sqrt{s}} - (1 \leftrightarrow 2) - (3 \leftrightarrow 4), \quad (2.129)$$

$$s \equiv X_{01} X_{02} X_{03} X_{04} X_{12} X_{34}. \quad (2.130)$$

Putting everything together, we have

$$\left(\frac{X_{14}}{X_{13}}\right)^{\frac{\Delta_{34}}{2}} \left(\frac{X_{24}}{X_{14}}\right)^{\frac{\Delta_{12}}{2}} \frac{g_{\Delta,\ell}^{\Delta_i}(u,v)}{X_{12}^{\frac{\Delta_1+\Delta_2}{2}} X_{34}^{\frac{\Delta_3+\Delta_4}{2}}} = \frac{1}{\mathcal{M}} \langle \phi_1(X_1) \phi_2(X_2) | J | \phi_3(X_3) \phi_4(X_4) \rangle \quad (2.131)$$

$$= \frac{\Gamma(\Delta + \ell)(2 - \Delta)}{\pi^2 \Gamma(\tilde{\Delta} + \ell - 1)} \frac{\Gamma\left(\frac{\tilde{\Delta} - \Delta_{34} + \ell}{2}\right) \Gamma\left(\frac{\tilde{\Delta} + \Delta_{34} + \ell}{2}\right)}{\Gamma\left(\frac{\Delta - \Delta_{34} + \ell}{2}\right) \Gamma\left(\frac{\Delta + \Delta_{34} + \ell}{2}\right)} X_{12}^{-\frac{\Delta_1 + \Delta_2 - \Delta}{2}} X_{34}^{-\frac{\Delta_3 + \Delta_4 - \tilde{\Delta}}{2}} \\ \times \int D^4 X_0 \frac{C_\ell^1(t)}{X_{10}^{\frac{\Delta + \Delta_{12}}{2}} X_{20}^{\frac{\Delta - \Delta_{12}}{2}} X_{30}^{\frac{\tilde{\Delta} + \Delta_{34}}{2}} X_{40}^{\frac{\tilde{\Delta} - \Delta_{34}}{2}}} \Big|_{M=1}. \quad (2.132)$$

Expanding the polynomial $C_\ell^1(t)$, the above integral becomes a sum of basic conformal four-point integrals (2.46) which can be expressed in terms of hypergeometric functions using (2.58). Luckily, the work of simplifying the resulting sum has already been performed in [64], using a recursion relation for Gegenbauer polynomials along with elementary hypergeometric function identities. The result is (2.9), which we reproduce here for the reader's convenience¹⁴

$$g_{\Delta,\ell}^{\Delta_i}(z, \bar{z}) = (-1)^\ell \frac{z \bar{z}}{z - \bar{z}} (k_{\Delta+\ell}(z) k_{\Delta-\ell-2}(\bar{z}) - (z \leftrightarrow \bar{z})) \\ k_\beta(x) \equiv x^{\beta/2} {}_2F_1\left(\frac{\beta - \Delta_{12}}{2}, \frac{\beta + \Delta_{34}}{2}, \beta, x\right). \quad (2.133)$$

It is not obvious from this derivation why (2.132) should telescope into such a compact form. We expect there should exist a simpler route to the correct answer, perhaps beginning by expressing the conformal integral over X_0 in twistor variables. This is clearly unnecessary in the case of conformal blocks for external scalars, since there the conformal Casimir equation can be solved directly (bypassing the calculation given here). However, it could prove helpful in simplifying expressions for higher spin conformal blocks. We leave further investigation of this idea to future work.

¹⁴Note that the $g_{\Delta,\ell}^{\Delta_i}$ quoted here differs by a factor of 2^ℓ from the one derived in [64]. This is a consequence of our two-point function normalization (2.118). We retain the factor of $(-1)^\ell$ because we have also chosen conventions where three-point function coefficients are real in unitary theories.

2.5.5 Example: Antisymmetric Tensor Exchange between Vectors

As an example that brings together all of the machinery in this section, let us consider the exchange of a self-dual antisymmetric tensor $F(X, S)$ (and its anti-self-dual conjugate $\overline{F}(X, \overline{S})$) in a four-point function of vectors $J_i(X_i, S_i, \overline{S}_i)$. This computation could also be performed using the tensor formalism of section 2.4.2, where the embedding space ϵ -tensor enters the self-duality condition for F . In twistor language, the three independent structures that can appear in the three-point function $\langle F J_1 J_2 \rangle$ are

$$\begin{aligned} \langle F(X_0, S_0) J_1(X_1, S_1, \overline{S}_1) J_2(X_2, S_2, \overline{S}_2) \rangle = & \lambda_0 \frac{I_{0\overline{1}} I_{0\overline{2}} K_{102}}{X_{01}^{\frac{\Delta+\Delta_1-\Delta_2+2}{2}} X_{02}^{\frac{\Delta+\Delta_2-\Delta_1+2}{2}} X_{12}^{\frac{\Delta_1+\Delta_2-\Delta}{2}}} \\ & + \lambda_1 \frac{I_{0\overline{1}} I_{1\overline{2}} K_{012}}{X_{01}^{\frac{\Delta+\Delta_1-\Delta_2+2}{2}} X_{02}^{\frac{\Delta+\Delta_2-\Delta_1}{2}} X_{12}^{\frac{\Delta_1+\Delta_2-\Delta+2}{2}}} \\ & + \lambda_2 \frac{I_{0\overline{2}} I_{2\overline{1}} K_{021}}{X_{01}^{\frac{\Delta+\Delta_1-\Delta_2}{2}} X_{02}^{\frac{\Delta+\Delta_2-\Delta_1+2}{2}} X_{12}^{\frac{\Delta_1+\Delta_2-\Delta+2}{2}}} \end{aligned} \quad (2.134)$$

$$= \sum_{i=0,1,2} \langle F J_1 J_2 \rangle^{(i)} \lambda_i. \quad (2.135)$$

Similarly, we have

$$\langle \overline{F} J_3 J_4 \rangle = \sum_{i=0,1,2} \langle \overline{F} J_3 J_4 \rangle^{(i)} \overline{\lambda}_i, \quad (2.136)$$

where the structures $\langle \overline{F} J_3 J_4 \rangle^{(i)}$ are obtained from $\langle F J_1 J_2 \rangle^{(i)}$ by replacing $1, 2 \rightarrow 3, 4$ and conjugating the spinor invariants $I_{i\overline{j}} \rightarrow I_{j\overline{i}}$, $K_{ijk} \rightarrow \overline{K}_{ijk}$.

The shadow transform of $\langle \overline{F} J_3 J_4 \rangle$ is given by

$$\langle \widetilde{\overline{F}} J_3 J_4 \rangle = \langle F J_3 J_4 \rangle^{(i)}|_{\Delta \rightarrow \widetilde{\Delta}} \mathcal{S}_i^j \overline{\lambda}_j, \quad (2.137)$$

where the structures $\langle F J_3 J_4 \rangle^{(i)}|_{\Delta \rightarrow \widetilde{\Delta}}$ are those appearing in (2.134) with the replacements $1, 2 \rightarrow$

3, 4, and $\Delta \rightarrow \tilde{\Delta}$, and the matrix \mathcal{S} has entries

$$\mathcal{S} = \frac{\pi^2 \Gamma(\Delta - 1)}{\Gamma(\tilde{\Delta} + 1)} \begin{pmatrix} -A_{1,1}^{1,1} & 0 & 0 \\ \frac{(\Delta - 2 + \Delta_{34})}{2} A_{1,1}^{1,0} & 0 & A_{0,1}^{1,0} \\ \frac{(2 - \Delta + \Delta_{34})}{2} A_{1,1}^{0,1} & A_{1,0}^{0,1} & 0 \end{pmatrix} \quad (2.138)$$

$$A_{l,k}^{m,n} \equiv \frac{\Gamma(m + \frac{\tilde{\Delta} - \Delta_{34}}{2}) \Gamma(n + \frac{\tilde{\Delta} + \Delta_{34}}{2})}{\Gamma(l + \frac{\Delta - \Delta_{34}}{2}) \Gamma(k + \frac{\Delta + \Delta_{34}}{2})}. \quad (2.139)$$

In deriving (2.137) and (2.138), we have used the relation

$$\overline{K}_{304} K_{034} K_{043} = K_{043} I_{04} \overline{I}_{43} X_{03} - K_{034} I_{03} \overline{I}_{34} X_{04} - K_{304} I_{03} \overline{I}_{04} X_{34}. \quad (2.140)$$

As before, we can compute the appropriate mixing matrix by iterating the shadow transform twice,

$$\begin{aligned} \langle \overline{F} J_3 J_4 \rangle^{(j)} \mathcal{M}_j^i \lambda_i &= \langle \overline{F} | F | J_3 J_4 \rangle \\ &= \langle \widetilde{\overline{F}} J_3 J_4 \rangle \\ &= \langle \overline{F} J_3 J_4 \rangle^{(k)} (\mathcal{S}|_{\Delta \rightarrow \tilde{\Delta}})_k^j \mathcal{S}_j^i \lambda_i \\ &= \frac{\pi^4}{\Delta(\Delta - 1)(\Delta - 3)(\Delta - 4)} \langle \overline{F} J_3 J_4 \rangle^{(i)} \lambda_i. \end{aligned} \quad (2.141)$$

Here, it turns out that \mathcal{M}_i^j is proportional to the identity matrix. This is in fact a general result for the exchange of any operator in a completely left-handed $(j/2, 0)$ or completely right-handed $(0, \bar{j}/2)$ representation of the Lorentz group.

The conformal partial wave for exchange of F in a four-point function of currents J_i is then given by

$$\begin{aligned} \lambda_i \overline{\lambda}_j \left(\frac{X_{14}}{X_{24}} \right)^{\frac{\Delta_{12}}{2}} \left(\frac{X_{13}}{X_{14}} \right)^{\frac{\Delta_{34}}{2}} \frac{g_F^{i,j}(X_i, S_i, \overline{S}_i)}{X_{12}^{\frac{\Delta_1 + \Delta_2}{2} + 1} X_{34}^{\frac{\Delta_3 + \Delta_4}{2} + 1}} \\ = \lambda_i^{(i)} \langle J_1 J_2 | F | J_3 J_4 \rangle^{(k)} (\mathcal{M}^{-1})_k^j \overline{\lambda}_j \end{aligned} \quad (2.142)$$

$$= \frac{\Delta(\Delta - 1)(\Delta - 3)(\Delta - 4)}{\pi^4} \frac{1}{2!^2} \int D^4 X \langle J_1 J_2 F(X, S) \rangle (\overline{\partial}_S X \overrightarrow{\partial}_T)^2 \langle \widetilde{\overline{F}}(X, T) J_3 J_4 \rangle. \quad (2.143)$$

This is a sum of tensor four-point integrals of the form (2.62), which can be evaluated using (2.65) and (2.58). The full 3×3 matrix of conformal blocks $g_F^{i,j}$ contains approximately a hundred terms, so for brevity we will present only a single component in the main text,

$$g_F^{1,1} = \frac{u^\Delta \Delta}{(\Delta-1)(\Delta-2)(\Delta+\Delta_{34})} \left[\begin{aligned} & I_{1\bar{2}} I_{2\bar{1}} I_{3\bar{4}} I_{4\bar{3}} \left(-F_{3,2;2}^{(4)} + \frac{(1+u-v)(\Delta-\Delta_{34})}{2} F_{2,2;1}^{(4)} - \frac{u(\Delta-\Delta_{34})(2+\Delta-\Delta_{34})}{4} F_{1,2;0}^{(4)} \right) \\ & - \frac{I_{1\bar{2}} I_{4\bar{3}} K_{213} \bar{K}_{134}}{X_{13}} F_{3,2;2}^{(4)} + I_{1\bar{2}} I_{4\bar{3}} \left(\frac{K_{243} \bar{K}_{134}}{X_{34}} + \frac{K_{213} \bar{K}_{124}}{X_{12}} \right) \frac{u(\Delta-\Delta_{34})}{2} F_{2,2;1}^{(4)} \\ & - \frac{I_{1\bar{2}} I_{4\bar{3}} K_{243} \bar{K}_{124}}{X_{24}} \frac{u(\Delta-\Delta_{34})(2+\Delta-\Delta_{34})}{2} F_{1,2;0}^{(4)} \end{aligned} \right], \quad (2.144)$$

where

$$F_{m,n;k}^{(h)}(z, \bar{z}) \equiv \frac{\left(\frac{\Delta-\Delta_{12}}{2}\right)_m \left(\frac{\Delta+\Delta_{34}}{2}\right)_n}{(\Delta)_k^2} \left(-\frac{\partial}{\partial v}\right)^{h-1} k_{m,n;k}(z) k_{m,n;k}(\bar{z}) \quad (2.145)$$

$$k_{m,n;k}(x) \equiv {}_2F_1\left(\frac{\Delta-\Delta_{12}}{2} - m, \frac{\Delta+\Delta_{34}}{2} - n, \Delta - k, x\right). \quad (2.146)$$

2.6 Discussion

The strategy for computing higher-spin conformal blocks is as follows:

1. Lift primary operators to the embedding space.
2. Find a gauge- and conformally-invariant projector $|\mathcal{O}|$.
3. Determine the proper normalization (or mixing matrix) by inserting $|\mathcal{O}|$ within a three-point function.
4. Conformal blocks are then given by inserting $|\mathcal{O}|$ within a four-point function. Perform the monodromy-projected conformal integrals using the formulae in section 2.3.

We have shown how to apply this strategy to tensor operators in d -dimensions, and arbitrary operators in 4d, where we introduced an efficient formalism for writing down conformally-invariant

correlators using auxiliary twistors. But it should apply equally well in any setting. In particular, it would be interesting to apply it to superconformal theories, perhaps providing a way to bypass the complicated superconformal block calculations in [57, 78, 79]. This would require generalizing the notion of conformal integrals to superconformal integrals. For 4d CFTs, the underlying twistor structure of the projective null-cone, and the superembedding formalism of [80] may play an important role. Efficient methods for computing superconformal blocks could be especially valuable in six dimensions, where the bootstrap might shed light on the mysterious $\mathcal{N} = (2, 0)$ M5 brane SCFT.

An important task for applying bootstrap methods to higher-spin operators is now to compute all conformal blocks that can appear in a given four-point function. For example, an OPE of currents $J_1^\mu \times J_2^\nu$ in four dimensions can contain any operator transforming in a Lorentz representation of the form $(\frac{j}{2}, \frac{j}{2}), (\frac{j+2}{2}, \frac{j}{2}), (\frac{j}{2}, \frac{j+2}{2}), (\frac{j+4}{2}, \frac{j}{2}), (\frac{j}{2}, \frac{j+4}{2}), j \geq 0$. The $(\frac{j}{2}, \frac{j}{2})$ operators are traceless symmetric tensors, and their conformal blocks can be derived easily using the methods of [68]. We have given as an example the computation for $(0, 1)$ and $(1, 0)$ operators. However, to apply bootstrap methods to a four-point function of currents, we need conformal blocks for *all* possible operators, so a formidable task is in store.¹⁵

Given the formulae in section 2.3, our methods are algorithmic and can be readily computerized. However, there is also reason to believe that compact analytic expressions might exist even for very general classes of conformal blocks. In particular, we have not shed light on why the terms in the conformal block for spin- ℓ exchange between external scalars can be combined into such a simple form (2.9). Dolan and Osborn understood this using the conformal Casimir equation. Although their argument becomes intractable in the case of higher spin, it's likely that similar structure is present.

¹⁵Experience has shown that it is sometimes sufficient to compute recursion relations which allow for efficient numerical computation and tabulation of conformal blocks and their derivatives [63], so completely general formulae are not obligatory.

Finally, let us note that the technology developed here should work equally well in Mellin space [81, 82], which is proving to be a convenient setting for understanding effective CFTs [83, 84] dual to weakly coupled theories in AdS [77, 85–87]. The only modification would be the expressions for conformal integrals (2.55, 2.58), which become functions of Mellin variables δ_{ij} instead of conformal cross-ratios. Higher spin conformal blocks in Mellin space could be useful for understanding the gauge and gravity sectors of effective CFTs.

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2.A Monodromy Projections and the OPE

The prescription (2.34) for ensuring the conformal block’s consistency with the OPE may seem somewhat ad hoc, so let us clarify why it is needed. Along the way, we will elucidate the origin of the shadow contribution $g_{\tilde{\mathcal{O}}}$. Recall that our “candidate” conformal block for the exchange of a dimension- Δ scalar \mathcal{O} in a four-point function of dimension- δ scalars ϕ is given by

$$F(X_i) \propto X_{12}^\delta X_{34}^\delta \int D^d X D^d Y \langle \phi(X_1) \phi(X_2) \mathcal{O}(X) \rangle \frac{1}{(-2X \cdot Y)^{d-\Delta}} \langle \mathcal{O}(Y) \phi(X_3) \phi(X_4) \rangle. \quad (2.147)$$

Why should (2.147) violate the OPE in the first place? This is clear already in the integrand: the objects $\langle \phi \phi \mathcal{O} \rangle$ are radially-ordered expectation values of fields. In any given quantization, they include pieces where the $\phi \times \phi$ OPE is valid, and pieces where it is invalid. For concreteness, restrict to the Poincaré section and consider radial quantization around the origin.

We may write

$$\langle \phi(x_1)\phi(x_2)\mathcal{O}(x) \rangle = \langle 0|\mathcal{R}\{\phi(x_1)\phi(x_2)\mathcal{O}(x)\}|0\rangle \quad (2.148)$$

$$\begin{aligned} &= \theta(|x| > |x_1|, |x_2|) \langle 0|\mathcal{O}(x)\mathcal{R}\{\phi(x_1)\phi(x_2)\}|0\rangle \\ &\quad + \theta(|x| < |x_1|, |x_2|) \langle 0|\mathcal{R}\{\phi(x_1)\phi(x_2)\}\mathcal{O}(x)|0\rangle \\ &\quad + \text{other orderings,} \end{aligned} \quad (2.149)$$

where $\mathcal{R}\{\dots\}$ indicates radial-ordering. In the first term, the $\phi \times \phi$ OPE is valid, since $\mathcal{O}(x)$ lies outside a sphere surrounding x_1, x_2 . However, the $\phi \times \phi$ OPE does not converge in the other terms.¹⁶

The different orderings in (2.149) are distinguished by their monodromy properties. Specifically, consider the transformation $M = e^{2\pi i(\mathcal{D}_1 + \mathcal{D}_2)}$, where $\mathcal{D} = x \cdot \frac{\partial}{\partial x}$ is the differential operator generating dilatations and \mathcal{D}_i indicates \mathcal{D} acting on the point x_i . Clearly, $Mx_{12}^2 = e^{4\pi i}x_{12}^2$, while $Mx_{ij}^2 = x_{ij}^2$ for all other pairs i, j , assuming x_3 and x_4 are far from the origin. If $\phi(x)$ is primary with dimension δ , we have

$$e^{\lambda(\mathcal{D} + \delta)}\phi(x) = e^{\lambda\mathcal{D}}\phi(x)e^{-\lambda\mathcal{D}}, \quad (2.150)$$

where D generates dilatations on the Hilbert space. Notice also that states $\mathcal{O}(x)|0\rangle$ have energies of the form $\Delta + n$, where $n \in \mathbb{Z}$ (the primary state $|\mathcal{O}\rangle$ has energy Δ , while descendants $P^{\mu_1} \dots P^{\mu_n}|\mathcal{O}\rangle$ have energy $\Delta + n$). Consequently,

$$e^{\pm 2\pi i D}\mathcal{O}(x)|0\rangle = \mathcal{O}(x)|0\rangle e^{\pm 2\pi i \Delta} \quad (2.151)$$

$$\langle 0|\mathcal{O}(x)e^{\pm 2\pi i D} = e^{\pm 2\pi i \Delta}\langle 0|\mathcal{O}(x). \quad (2.152)$$

Applying M to the radially-ordered correlator and using these facts, each ordering picks up a

¹⁶Of course, for a three-point function we can restore validity of the OPE by quantizing around a different point. However, no single point ensures validity of the OPE for all values of x_1, x_2 , and x .

different phase

$$\begin{aligned}
e^{2\pi i(\mathcal{D}_1+\mathcal{D}_2)}\langle\phi(x_1)\phi(x_2)\mathcal{O}(x)\rangle &= \theta(|x| > |x_1|, |x_2|)\langle 0|\mathcal{O}(x)\mathcal{R}\{\phi(x_1)\phi(x_2)\}|0\rangle e^{2\pi i(\Delta-2\delta)} \\
&+ \theta(|x| < |x_1|, |x_2|)\langle 0|\mathcal{R}\{\phi(x_1)\phi(x_2)\}\mathcal{O}(x)|0\rangle e^{2\pi i(-\Delta-2\delta)} \\
&+ \text{other orderings.}
\end{aligned} \tag{2.153}$$

We see that the $\langle 0|\mathcal{O}\phi\phi|0\rangle$ ordering, where the OPE is valid, contributes precisely to the part of $F(X_i)$ with monodromy $e^{2\pi i\Delta}$, namely $g_{\mathcal{O}}$. (When acting on $F(X_i)$ the phases $e^{2\pi i(-2\delta)}$ are cancelled by the factor $x_{12}^{2\delta}$ out front.) Meanwhile, the $\langle 0|\phi\phi\mathcal{O}|0\rangle$ ordering contributes to the shadow block $g_{\tilde{\mathcal{O}}}$ (assuming $d \in \mathbb{Z}$). Thus, projection onto the correct monodromy eigenspace is equivalent to including the θ -functions $\theta(|x| > |x_1|, |x_2|)$ in the integral (2.29), carving out a sphere around x_1, x_2 and ensuring validity of the OPE.

The appearance of θ -functions in the integrand raises a puzzle. The form of these θ -functions depends on our choice of dilatation operator \mathcal{D} , since different choices imply different radial orderings. Thus, they naïvely break conformal invariance. However, the monodromy argument makes it clear that this breaking is somehow weak. Monodromy projection introduces similar θ -functions in the conformal block (but they take the value 1 when $|x_{1,2}| \ll |x_{3,4}|$ so we have ignored them in the main text). Somehow, changing the θ -functions in the integrand changes *only* these θ -functions in the result. It would be interesting to understand why in more detail.

2.B Spinor Conventions in Six Dimensions

We choose $-++$ signature for the metric $g_{\mu\nu}$ in 4d Minkowski space, and follow the conventions of Wess and Bagger [88] for four-dimensional spinors. The six-dimensional embedding space metric is given by $\eta_{mn}X^mX^n = -X^+X^- + g_{\mu\nu}X^\mu X^\nu$.

Six-dimensional spinors (twistors) decompose under the 4d Lorentz group as

$$Z_A = \begin{pmatrix} \lambda_\alpha \\ \mu^{\dot{\alpha}} \end{pmatrix}. \quad (2.154)$$

We choose conventions where the $SU(2, 2)$ -invariant antisymmetric tensors $\epsilon_{ABCD}, \epsilon^{ABCD}$ satisfy $\epsilon_{1234} = \epsilon^{1234} = +1$. We have the antisymmetric chiral Gamma matrices

$$\Gamma_{AB}^+ = \begin{pmatrix} 0 & 0 \\ 0 & 2i\epsilon^{\dot{\alpha}\dot{\beta}} \end{pmatrix}, \quad \Gamma_{AB}^- = \begin{pmatrix} -2i\epsilon_{\alpha\beta} & 0 \\ 0 & 0 \end{pmatrix}, \quad \Gamma_{AB}^\mu = \begin{pmatrix} 0 & \sigma_{\alpha\dot{\gamma}}^\mu \epsilon^{\dot{\gamma}\dot{\beta}} \\ -\bar{\sigma}^{\mu\dot{\alpha}\gamma} \epsilon_{\gamma\beta} & 0 \end{pmatrix}. \quad (2.155)$$

And also $\tilde{\Gamma}^{mAB} = \frac{1}{2}\epsilon^{ABCD}\Gamma_{CD}^m$, which are given by

$$\tilde{\Gamma}^{+AB} = \begin{pmatrix} 2i\epsilon^{\alpha\beta} & 0 \\ 0 & 0 \end{pmatrix}, \quad \tilde{\Gamma}^{-AB} = \begin{pmatrix} 0 & 0 \\ 0 & -2i\epsilon_{\dot{\alpha}\dot{\beta}} \end{pmatrix}, \quad \tilde{\Gamma}^{\mu AB} = \begin{pmatrix} 0 & -\epsilon^{\alpha\gamma}\sigma_{\gamma\dot{\beta}}^\mu \\ \epsilon_{\dot{\alpha}\dot{\gamma}}\bar{\sigma}^{\mu\dot{\gamma}\beta} & 0 \end{pmatrix}. \quad (2.156)$$

They satisfy

$$(\Gamma^m \tilde{\Gamma}^n + \Gamma^n \tilde{\Gamma}^m)_A{}^B = -2\eta^{nm} \delta_A{}^B \quad (2.157)$$

$$(\tilde{\Gamma}^m \Gamma^n + \tilde{\Gamma}^n \Gamma^m)^A{}_B = -2\eta^{nm} \delta^A{}_B \quad (2.158)$$

$$\tilde{\Gamma}^{mAB} \Gamma_{mCD} = 2(\delta_C^A \delta_D^B - \delta_C^B \delta_D^A) \quad (2.159)$$

$$\tilde{\Gamma}^{mAB} \tilde{\Gamma}_m{}^{CD} = 2\epsilon^{ABCD} \quad (2.160)$$

$$\Gamma_{AB}^m \Gamma_{mCD} = 2\epsilon_{ABCD}. \quad (2.161)$$

With them, we can define bi-spinors

$$\bar{X}^{AB} = X_m \tilde{\Gamma}^{mAB}, \quad X_{AB} = X_m \Gamma_{AB}^m. \quad (2.162)$$

The inverse transformation is

$$X^m = \frac{1}{4} X_{AB} \tilde{\Gamma}^{mAB} = \frac{1}{4} \bar{X}^{AB} \Gamma_{AB}^m. \quad (2.163)$$

We also have the inner product

$$X^m Y_m = -\frac{1}{4} \text{Tr}(\overline{X} Y). \quad (2.164)$$

Chapter 3

Superconformal Blocks

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3.1 Superconformal Blocks

At this stage we could proceed to derive bounds on operator dimensions and OPE coefficients in general CFTs. However, because we would also like to derive similar bounds in $\mathcal{N} = 1$ superconformal theories, we will first consider more carefully the additional constraints imposed by supersymmetry. In particular, three-point functions of primary operators in the same supersymmetry multiplet are related to each other by the superconformal algebra, and one can construct “superconformal blocks” which sum up the contributions of all operators in a given superconformal multiplet.

We will focus on four-point functions involving a complex scalar ϕ that is the lowest component of a chiral superfield Φ of dimension $d = \frac{3}{2}R_\Phi$. In terms of the operators appearing in

the $\phi \times \phi^*$ OPE, the superconformal block decomposition looks like

$$\langle \phi(x_1) \phi^*(x_2) \phi(x_3) \phi^*(x_4) \rangle = \frac{1}{x_{12}^{2d} x_{34}^{2d}} \sum_{\mathcal{O} \in \Phi \times \Phi^\dagger} |\lambda_{\mathcal{O}}|^2 (-1)^l \mathcal{G}_{\Delta,l}(u, v). \quad (3.1)$$

Here, we have adopted the notation $\mathcal{O} \in \Phi \times \Phi^\dagger$ to indicate that the sum is over *superconformal* primaries \mathcal{O} appearing in $\phi \times \phi^*$, and not simply primaries under the conformal subgroup. By definition, superconformal primary operators \mathcal{O} are annihilated by the S and \bar{S} generators in the superconformal algebra, from which it follows that they are also annihilated by the K generator. However, a finite number of superconformal descendants of \mathcal{O} are also killed by K , so one may decompose $\mathcal{G}_{\Delta,l}(u, v)$ into a finite sum of conformal blocks $g_{\Delta,l}(u, v)$.

Just as the explicit expression (2.9) for conformal blocks was crucial for the analysis of [54–56], an explicit expression for superconformal blocks will be crucial for us. We find that $\mathcal{N} = 1$ superconformal blocks in the $\phi \times \phi^*$ channel are given by

$$\begin{aligned} \mathcal{G}_{\Delta,l} = & g_{\Delta,l} - \frac{(\Delta + l)}{2(\Delta + l + 1)} g_{\Delta+1,l+1} - \frac{(\Delta - l - 2)}{8(\Delta - l - 1)} g_{\Delta+1,l-1} \\ & + \frac{(\Delta + l)(\Delta - l - 2)}{16(\Delta + l + 1)(\Delta - l - 1)} g_{\Delta+2,l}, \end{aligned} \quad (3.2)$$

where we will take conformal blocks in this chapter have a different normalization from in the previous chapter,

$$\begin{aligned} g_{\Delta,l}(z, \bar{z}) &= \frac{(-1)^l}{2^l} \frac{z \bar{z}}{z - \bar{z}} (k_{\Delta+l}(z) k_{\Delta-l-2}(\bar{z}) - z \leftrightarrow \bar{z}) \\ k_\beta(x) &\equiv x^{\beta/2} {}_2F_1(\beta/2, \beta/2, \beta; x). \end{aligned} \quad (3.3)$$

To our knowledge, this expression has not yet appeared in the literature, though analogous results for $\mathcal{N} = 2$ and $\mathcal{N} = 4$ theories are known [78]. Eq. (3.2) is the key ingredient we need to apply the technology of Section 4.2.4 to superconformal theories. In the following subsections, we will give two derivations — one involving explicit analysis of superconformal two- and three-point functions, and another quicker but less illuminating argument leveraging known expressions from $\mathcal{N} = 2$ theories

[78]. The discussion is somewhat technical, and readers interested solely in bounds on dimensions and OPE coefficients should feel free to skip to Section 5.5.

Our first derivation of Eq. (3.2) proceeds as follows. We start by understanding which superconformal primary operators $\mathcal{O}^{a_1 \dots a_l}$ can appear in the OPE $\phi \times \phi^*$. We then determine which superconformal descendants of $\mathcal{O}^{a_1 \dots a_l}$ are conformal primaries, and further calculate the relationships between two- and three- point functions of these conformal primaries. Since each conformal primary contributes a block $g_{\Delta', l'}$ to $\langle \phi \phi^* \phi \phi^* \rangle$, we can piece together $\mathcal{G}_{\Delta, l}$ from these contributions. For completeness we also include a brief discussion of the $\phi \times \phi$ channel. However, in this case only a single operator in each supersymmetry multiplet may contribute, so the superconformal blocks turn out to be the same as the conformal blocks Eq. (3.3). Our conventions for the superconformal algebra and spinor notation are summarized in Appendix 3.A.

3.1.1 Superconformal Three-Point Functions

$\phi \times \phi$ OPE

Let us start by examining the $\phi \times \phi$ OPE, since the constraints from superconformal symmetry are particularly transparent in this case. This analysis is not needed later, but we include it for completeness and to establish some notation. For some previous discussions of this OPE, see [89, 90]. In this subsection we will follow the notation and conventions of [91], where a superconformal primary \mathcal{O}^I (I denotes Lorentz indices) is specified by spins (j, \bar{j}) and conformal weights $(q_{\mathcal{O}}, \bar{q}_{\mathcal{O}})$, which are related to the dimension and R -charge via $q_{\mathcal{O}} + \bar{q}_{\mathcal{O}} = \Delta_{\mathcal{O}}$ and $\frac{2}{3}(q_{\mathcal{O}} - \bar{q}_{\mathcal{O}}) = R_{\mathcal{O}}$. The unitarity bound for non-chiral superconformal primary operators then requires [92–94]

$$\Delta_{\mathcal{O}^I} \geq \left\lfloor \frac{3}{2} R_{\mathcal{O}^I} - j + \bar{j} \right\rfloor + j + \bar{j} + 2. \quad (3.4)$$

To begin, note that since $\bar{Q}\phi(x) = 0$, only operators that are annihilated by \bar{Q} may appear

in $\phi \times \phi$. A priori, there are four possibilities:

1. Chiral primaries. Since these transform in $(j, 0)$ representations of the Lorentz group $SU(2) \times SU(2)$, they can appear only if $j = 0$. We will denote the linear combination of chiral primaries appearing in $\phi \times \phi$ by ϕ^2 .
2. Descendants of the form $\overline{Q}^{(\dot{\alpha}_1} \mathcal{O}^{\dot{\alpha}_2 \dots \dot{\alpha}_l) \alpha_1 \dots \alpha_l}$, where l is even and \mathcal{O}^I satisfies the shortening condition

$$\overline{Q}_{\dot{\alpha}} \mathcal{O}^{\dot{\alpha} \dot{\alpha}_3 \dots \dot{\alpha}_l \alpha_1 \dots \alpha_l} = 0. \quad (3.5)$$

(Note that this implies $\overline{Q}^2 \mathcal{O}^I = 0$, so that $\overline{Q} \mathcal{O}^I$ is indeed killed by \overline{Q} .) The superconformal algebra implies [91] that such operators satisfy $\overline{q}_{\mathcal{O}} = (l+1)/2$. Then using $R_{\overline{Q} \mathcal{O}^I} = 2R_{\Phi}$ we find $\Delta_{\mathcal{O}^I} = 2d + l - \frac{1}{2}$, so that the dimensions of these operators are determined by their spins. We will denote the linear combination of these descendants with spin l as $\overline{Q} \mathcal{O}_l^I$.¹

3. Descendants of the form $\overline{Q}_{\dot{\alpha}} \mathcal{O}^{\dot{\alpha} \dot{\alpha}_1 \dots \dot{\alpha}_l \alpha_1 \dots \alpha_l}$, where \mathcal{O} satisfies the shortening condition

$$\overline{Q}^{(\dot{\alpha}_1} \mathcal{O}^{\dot{\alpha}_2 \dots \dot{\alpha}_{l+2}) \alpha_1 \dots \alpha_l} = 0. \quad (3.6)$$

Such multiplets must satisfy $\overline{q}_{\mathcal{O}} = -(l+1)/2$, which implies upon matching R -charges that $\Delta_{\mathcal{O}} = 2d - l - 5/2$. However, this violates the unitarity bound Eq. (3.4), so such operators actually cannot appear.

4. Descendants of the form $\overline{Q}^2 Q^{2-n} \mathcal{O}^I$, with $n = 0, 1, 2$.

Thus, we expect the OPE to take the form

$$\phi(x)\phi(0) = C(x, P)\phi^2(0) + \sum_{l=2,4,\dots} C_I^l(x, P) \overline{Q} \mathcal{O}_l^I(0) + \sum_{\mathcal{O}^I} \overline{Q}^2 C_I(x, P, Q) \mathcal{O}^I(0), \quad (3.7)$$

¹We are grateful to Alessandro Vichi for pointing out the possibility of these operators in the $\phi \times \phi$ OPE.

where the latter sum runs over superconformal primaries with $R_{\mathcal{O}^I} = 2R_\Phi - n$, and a priori $n = 0, 1, 2$ depending on how many powers of Q appear.

We can obtain additional constraints on the operators \mathcal{O}^I by acting on both sides of Eq. (3.7) with an S generator. Note that S kills the left-hand side because $[S, P] \sim \bar{Q}$ and ϕ is chiral and primary. On the right-hand side, we can commute S through all powers of \bar{Q} and P , since $\{S, \bar{Q}\} = 0$ and $\bar{Q}^2[S, P] \sim \bar{Q}^3 = 0$. However, if powers of Q were present, there would be terms involving $\{S, Q\}$ which would not vanish when acting on \mathcal{O}^I . Thus, we conclude that $C_I(x, P, Q) = C_I(x, P)$ and therefore $R_{\mathcal{O}^I} = 2R_\Phi - 2$. In this case the I indices must correspond to even-spin operators due to the symmetry under exchanging $x \leftrightarrow -x$. Finally, the unitarity bound Eq. (3.4) implies $\Delta_{\bar{Q}^2 \mathcal{O}^I} \geq |3R_\Phi - 3| + l + 3$. Note also that $\bar{Q}^2 \mathcal{O}^I$ is primary under the conformal sub-algebra.

Instead of playing directly with the superconformal generators, an alternative approach that will prove useful later is to consider the general form of superconformal-covariant three-point functions. Let us take a moment to recover the above results using this language.

The $\Phi \times \Phi$ OPE contains a superconformal multiplet \mathcal{O}^I if and only if the three-point function $\langle \Phi(z_{1+}) \Phi(z_{2+}) \mathcal{O}^{I\dagger}(z_3) \rangle$ is non-vanishing, where the z 's are superspace coordinates $(x, \theta, \bar{\theta})$, and z_+ indicates dependence only on the chiral subspace $(x + i\theta\sigma\bar{\theta}, \theta)$. The general form of such a three-point function consistent with superconformal symmetry is

$$\langle \Phi(z_{1+}) \Phi(z_{2+}) \mathcal{O}^{I\dagger}(z_3) \rangle = \frac{t^I(X_3, \Theta_3, \bar{\Theta}_3)}{x_{31}^{2d} x_{32}^{2d}}, \quad (3.8)$$

where $x_{ij}^- = x_{i-} + 2i\theta_j\sigma\bar{\theta}_i - x_{j+}$ denotes the supertranslation-invariant interval built out of anti-/chiral coordinates $x_{i\pm} = x_i \pm i\theta_i\sigma\bar{\theta}_i$, X_3 and Θ_3 are given by

$$X_3^a = -\frac{1}{2} \frac{x_{31}^b x_{12}^c x_{23}^d}{x_{13}^2 x_{32}^2} \text{tr}(\bar{\sigma}^a \sigma_b \bar{\sigma}_c \sigma_d), \quad (3.9)$$

$$\Theta_3 = i \frac{x_{31}^a}{x_{13}^2} \sigma_a \bar{\theta}_{31} - i \frac{x_{32}^a}{x_{23}^2} \sigma_a \bar{\theta}_{32}, \quad \bar{\Theta}_3 = \Theta_3^\dagger, \quad (3.10)$$

and t^I has the homogeneity properties

$$t^I(\lambda\bar{\lambda}X_3, \lambda\Theta_3, \bar{\lambda}\bar{\Theta}_3) = \lambda^{2a}\bar{\lambda}^{2\bar{a}}t^I(X_3, \Theta_3, \bar{\Theta}_3) \quad (3.11)$$

with $a = \frac{1}{3}(2q_{\mathcal{O}} + \bar{q}_{\mathcal{O}} - 4d)$ and $\bar{a} = \frac{1}{3}(q_{\mathcal{O}} + 2\bar{q}_{\mathcal{O}} - 2d)$.

Since the covariant derivative $\bar{D}_1^{\dot{\alpha}}$ vanishes when acting on the left hand side of Eq. (3.8), we obtain an additional constraint (using Eqs. (6.1) and (6.2) in [91])

$$\begin{aligned} 0 &= \bar{D}_1^{\dot{\alpha}}t^I(X_3, \Theta_3, \bar{\Theta}_3) \\ &= -i\frac{(x_{\bar{1}3})^{\dot{\alpha}\alpha}}{x_{\bar{3}1}^2}\left(\frac{\partial}{\partial\Theta_3^\alpha} - 2i(\sigma^a\bar{\Theta}_3)_\alpha\frac{\partial}{\partial X_3^a}\right)t^I(X_3, \Theta_3, \bar{\Theta}_3), \end{aligned} \quad (3.12)$$

which implies that $t^I(X_3, \Theta_3, \bar{\Theta}_3) = t^I(\bar{X}_3, \bar{\Theta}_3)$, where $\bar{X}_3 \equiv X_3 + 2i\Theta_3\sigma\bar{\Theta}_3$. Finally, under $z_1 \leftrightarrow z_2$ we have $X_3 \leftrightarrow -\bar{X}_3$ and $\bar{\Theta}_3 \leftrightarrow -\bar{\Theta}_3$. There are three possible solutions to these constraints,

$$t^I(\bar{X}_3, \bar{\Theta}_3) = \text{const.}, \quad (3.13)$$

corresponding to \mathcal{O}^I being a chiral “ Φ^2 ” operator with $R_{\mathcal{O}} = 2R_{\Phi}$,

$$t^I(\bar{X}_3, \bar{\Theta}_3) \propto \bar{\Theta}_3^{(\dot{\alpha}_1}\bar{X}_3^{\dot{\alpha}_2} \dots \bar{X}_3^{\dot{\alpha}_l)} = \bar{\Theta}_3^{(\dot{\alpha}_1}X_3^{\dot{\alpha}_2} \dots X_3^{\dot{\alpha}_l)}, \quad (3.14)$$

corresponding to the short operators \mathcal{O}_l^I , and

$$t^I(\bar{X}_3, \bar{\Theta}_3) \propto \bar{\Theta}_3^2\bar{X}_3^{\Delta_{\mathcal{O}}-2d-l-1}\bar{X}_3^{a_1} \dots \bar{X}_3^{a_l} = \bar{\Theta}_3^2X_3^{\Delta_{\mathcal{O}}-2d-l-1}X_3^{a_1} \dots X_3^{a_l}, \quad (3.15)$$

corresponding to \mathcal{O}^I being a non-chiral operator with $R_{\mathcal{O}} = 2R_{\Phi} - 2$. Since the only irreducible Lorentz representations that can be built out of a single vector \bar{X}_3^a (or X_3^a) are traceless symmetric tensors, $\mathcal{O}^I = \mathcal{O}^{a_1 \dots a_l}$ must have definite integer spin $l = 2j = 2\bar{j}$, and invariance under $z_1 \leftrightarrow z_2$ further tells us that l must be even. The descendant operator $\bar{\mathcal{Q}}^2\mathcal{O}^I$ then has the correct quantum numbers to appear in the $\phi \times \phi$ OPE, in precise agreement with the preceding argument.

Here we see that for each supermultiplet appearing in $\Phi \times \Phi$, there is exactly one conformal primary appearing in $\phi \times \phi$. This is essentially because ϕ^2 , $\bar{\mathcal{Q}}\mathcal{O}_l^I$, and $\bar{\mathcal{Q}}^2\mathcal{O}^I$ are the only

conformal primaries in their respective supermultiplets with the correct R -charge. Consequently, the superconformal blocks for decomposing $\langle \phi \phi^* \phi \phi^* \rangle$ in the $\phi \times \phi$ channel are the same as the conformal blocks. Next we will turn to considering the $\phi \times \phi^*$ channel, where this will no longer be the case.

$\phi \times \phi^*$ OPE

We determine which operators can appear in the $\phi \times \phi^*$ OPE by examining three-point functions $\langle \Phi \Phi^\dagger \mathcal{O}^{I\dagger} \rangle$. Once again, let \mathcal{O}^I be a superconformal primary with conformal weights $(q_{\mathcal{O}}, \bar{q}_{\mathcal{O}})$ and spins (j, \bar{j}) . Following [91], we must have

$$\langle \Phi(z_{1+}) \Phi^\dagger(z_{2-}) \mathcal{O}^{I\dagger}(z_3) \rangle \propto \frac{1}{x_{31}^{2d} x_{23}^{2d}} t^I(X_3, \Theta_3, \bar{\Theta}_3), \quad (3.16)$$

where t^I satisfies Eq. (3.11) with $a = \frac{1}{3}(2q_{\mathcal{O}} + \bar{q}_{\mathcal{O}}) - d$ and $\bar{a} = \frac{1}{3}(2\bar{q}_{\mathcal{O}} + q_{\mathcal{O}}) - d$.

Demanding the appropriate chirality properties imposes further constraints. Just as in the $\phi \times \phi$ case, requiring $\bar{D}_1^{\dot{\alpha}} t^I = 0$ means t^I must be a function of \bar{X}_3 and $\bar{\Theta}_3$. We must additionally require

$$0 = D_2^\alpha t^I(\bar{X}_3, \bar{\Theta}_3) = i \frac{x_{23}^{\dot{\alpha}\alpha}}{x_{23}^2} \frac{\partial}{\partial \bar{\Theta}_3^\alpha} t^I(\bar{X}_3, \bar{\Theta}_3), \quad (3.17)$$

so that t^I is actually a function of \bar{X}_3 alone. Note that since the R -charge of \bar{X}_3 vanishes, the R -charge of the correlator $\langle \Phi \Phi^\dagger \mathcal{O}^{I\dagger} \rangle$ must vanish as well, which means $\mathcal{O}^I = \mathcal{O}^{I\dagger}$ should be a real operator with $q_{\mathcal{O}} = \bar{q}_{\mathcal{O}}$. Since we again can only build Lorentz representations out of a single vector \bar{X}_3^a , the only possibilities are traceless symmetric tensors, so $\mathcal{O}^I = \mathcal{O}^{a_1 \dots a_l}$ must have definite integer spin $l = 2j = 2\bar{j}$.

In summary, we have found that the only superconformal primaries appearing in the $\Phi \times \Phi^\dagger$ OPE are traceless symmetric tensors $\mathcal{O}^{a_1 \dots a_l}$ with vanishing R -charge. Superconformal symmetry determines the 3-point function to be

$$\langle \Phi(z_{1+}) \Phi^\dagger(z_{2-}) \mathcal{O}^{a_1 \dots a_l}(z_3) \rangle \propto \frac{1}{x_{31}^{2d} x_{23}^{2d}} \bar{X}_3^{\Delta_{\mathcal{O}} - 2d - l} \bar{X}_3^{a_1} \dots \bar{X}_3^{a_l} - \text{traces}. \quad (3.18)$$

In this case, the unitarity bound Eq. (3.4) requires $\Delta_{\mathcal{O}} \geq l + 2$.² The operators which can enter the OPE of the lowest components $\phi \times \phi^*$ are then R -charge zero descendants of a real superconformal primary, $P^n(Q\bar{Q})^m \mathcal{O}^{a_1 \dots a_l}$. To understand how these operators contribute to the four-point function $\langle \phi \phi^* \phi \phi^* \rangle$, we must now organize them into representations of the conformal sub-algebra.

3.1.2 Decomposition of Superconformal Multiplets into Conformal Multiplets

In this section, we will examine the structure of a multiplet built from a real superconformal primary $\mathcal{O}^{a_1 \dots a_l}$ of dimension Δ . The full superconformal multiplet can be decomposed into a direct sum of conformal multiplets, connected together by supersymmetry transformations. Here we will show explicitly how this decomposition works for operators that appear in the $\phi \times \phi^*$ OPE — namely operators of vanishing R -charge and definite spin. As a result, we will see how superconformal symmetry relates the OPE coefficients of different conformal primaries, and consequently how $\mathcal{G}_{\Delta, l}$ decomposes into a sum of $g_{\Delta, l}$'s.

Note that $\mathcal{O}^{a_1 \dots a_l}$ is symmetric and traceless in its indices. Throughout this subsection, we will adopt the convention of implicitly symmetrizing and subtracting traces in a_i for $i = 1, \dots, l$. This has the virtue of greatly simplifying notation, though one must be careful when manipulating expressions.

A convenient way to describe the descendants of a superconformal primary operator $\mathcal{O}^{a_1 \dots a_l}(0)$ is through superspace. For example, defining the superfield $\mathcal{O}^{a_1 \dots a_l}(x, \theta, \bar{\theta})$ by the action $e^{xP + \theta Q + \bar{\theta} \bar{Q}} \mathcal{O}^{a_1 \dots a_l}(0)$, we have the component expansion

$$\mathcal{O}^{a_1 \dots a_l}(x, \theta, \bar{\theta}) = A^{a_1 \dots a_l}(x) + \zeta_a B^{a a_1 \dots a_l}(x) + \zeta^2 D^{a_1 \dots a_l}(x) + \dots \quad (3.19)$$

where $\zeta_a \equiv \theta \sigma_a \bar{\theta}$, and “...” represents fields with non-zero R -charges. The component fields

²With an exception, of course, for the unit operator which has $\Delta = l = 0$.

$B^{aa_1\dots a_l}$ and $D^{a_1\dots a_l}$ are then related to $A^{a_1\dots a_l}$ through the action of Q and \overline{Q} as

$$B^{aa_1\dots a_l} = -\frac{1}{4}\Xi^a A^{a_1\dots a_l}, \quad (3.20)$$

$$D^{a_1\dots a_l} = -\frac{1}{64}\Xi_a B^{aa_1\dots a_l} - \frac{1}{16}\partial^2 A^{a_1\dots a_l}, \quad (3.21)$$

where we have defined $\Xi^a \equiv \overline{\sigma}^{a\dot{\alpha}\alpha}[Q_\alpha, \overline{Q}_{\dot{\alpha}}]$.

Both $A^{a_1\dots a_l}$ and $D^{a_1\dots a_l}$ are in the spin- l representation of the Lorentz group, but $B^{aa_1\dots a_l}$ can be further decomposed into irreducible representations. Recall that under $\text{SO}(4) \cong \text{SU}(2) \times \text{SU}(2)$, the spin- l representation of $\text{SO}(4)$ transforms as (j, j) with $j = l/2$. Since $B^{aa_1\dots a_l}$ has an additional vector index, it transforms as

$$\begin{aligned} (1/2, 1/2) \otimes (j, j) &= (j+1/2, j+1/2) \oplus (j-1/2, j-1/2) \\ &\oplus (j+1/2, j-1/2) \oplus (j-1/2, j+1/2). \end{aligned} \quad (3.22)$$

The first two components on the right-hand side are a spin- $(l+1)$ representation $J^{aa_1\dots a_l} \equiv B^{(aa_1\dots a_l)} - \text{traces}$, and a spin- $(l-1)$ representation $N^{a_2\dots a_l} \equiv B_b{}^{ba_1\dots a_l}$. The remaining two components comprise an operator $L^{aa_1\dots a_l}$ which is traceless and has vanishing total symmetrization. L can be further decomposed into irreducibles by projecting onto its “anti/self-dual” parts, satisfying $L_\pm^{aa_1\dots a_l} = \pm i \frac{l}{l+1} \epsilon^{aa_1}{}_{bc} L_\pm^{bca_2\dots a_l}$ (although this will not be important in our discussion). Notice that since L is not in a traceless symmetric representation, a primary operator built from it cannot appear in the OPE of ϕ with ϕ^* . Nonetheless, it will play a role in the identification of conformal primaries below. Altogether, we may write

$$B^{aa_1\dots a_l} = J^{aa_1\dots a_l} + \frac{l^2}{(l+1)^2} \eta^{aa_1} N^{a_2\dots a_l} + L^{aa_1\dots a_l}, \quad (3.23)$$

where as usual we are implicitly symmetrizing and subtracting traces in the a_i . The coefficient of N is such that the projection $N^{a_2\dots a_l} = B_b{}^{ba_1\dots a_l}$ works correctly.

Now let us consider the action of a special conformal generator K_a on the components of \mathcal{O} . We will be interested in determining which linear combinations of superconformal descendants

are annihilated by K_a . After some algebra, one can determine the action

$$K_a \begin{pmatrix} B^{ba_1 \dots a_l} \\ \epsilon^{ba_1}_{cd} P^c A^{da_2 \dots a_l} \end{pmatrix} = \begin{pmatrix} 2l \\ 2(\Delta - 1) \end{pmatrix} \begin{pmatrix} \epsilon^{ba_1}_{ad} A^{da_2 \dots a_l} \end{pmatrix}, \quad (3.24)$$

as well as

$$K_a \begin{pmatrix} D^{a_1 \dots a_l} \\ P^2 A^{a_1 \dots a_l} \\ P_b P^{a_1} A^{ba_2 \dots a_l} \\ \epsilon^{a_1 b}_{cd} P_b B^{cda_2 \dots a_l} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{l}{2} \\ 4(\Delta - 1) & -4l & 4l & 0 \\ 0 & 2(\Delta - l - 2) & 2(\Delta + l) & 0 \\ 2(l + 1) & -2(l - 1) & -2(l + 1) & 2(\Delta - 1) \end{pmatrix} \begin{pmatrix} P_a A^{a_1 \dots a_l} \\ P^{a_1} A_a^{a_2 \dots a_l} \\ \delta_a^{a_1} P_b A^{ba_2 \dots a_l} \\ \epsilon^{a_1}_{acd} B^{cda_2 \dots a_l} \end{pmatrix}, \quad (3.25)$$

from which we find that the linear combinations

$$B_{\text{prim}}^{aa_1 \dots a_l} \equiv B^{aa_1 \dots a_l} - \frac{l}{\Delta - 1} \epsilon^{aa_1}_{cd} P^c A^{da_2 \dots a_l} \quad (3.26)$$

$$D_{\text{prim}}^{a_1 \dots a_l} \equiv D^{a_1 \dots a_l} + \frac{l(l + 1) - (\Delta - 1)}{8(\Delta - 1)^2} P^2 A^{a_1 \dots a_l} - \frac{l^2}{4(\Delta - 1)^2} P_b P^{a_1} A^{ba_2 \dots a_l} - \frac{l}{4(\Delta - 1)} \epsilon^{a_1 b}_{cd} P_b B^{cda_2 \dots a_l} \quad (3.27)$$

are primary operators under the conformal subgroup. Note that only the L component of B is shifted in the above expression for B_{prim} , so that J and N are already primary.

An important fact is that when the unitarity bound $\Delta \geq l + 2$ is saturated, our superconformal multiplet is “shortened,” and the descendants N , L_{prim} , and D_{prim} actually vanish. For example, the supercurrent $\mathcal{J}^a(z)$ with $\Delta = 3$ and $l = 1$ contains only the R -symmetry current $J_R^a(x)$ and stress tensor $T^{ab}(x)$ as conformal primary components with vanishing R -charge. This will be reflected in explicit calculations below.

3.1.3 Conformal Primary Three-Point Functions

Next we would like to see how the three point functions $\langle \phi \phi^* J \rangle$, $\langle \phi \phi^* N \rangle$, and $\langle \phi \phi^* D_{\text{prim}} \rangle$ are related to $\langle \phi \phi^* A \rangle$. We will also verify that $\langle \phi \phi^* L_{\text{prim}} \rangle = 0$, as expected because L_{prim} is not in

an integer-spin (traceless symmetric) representation of the Lorentz group.

Let us set $\theta_1 = \theta_2 = \bar{\theta}_1 = \bar{\theta}_2 = 0$, and $\theta_3 = \theta, \bar{\theta}_3 = \bar{\theta}$ in the correlator Eq. (3.18) to get the 3-point function $\langle \phi(x_1) \phi^*(x_2) \mathcal{O}^{a_1 \dots a_l}(x_3, \theta, \bar{\theta}) \rangle$. Next, expanding in $\theta, \bar{\theta}$ and comparing with our component expansion Eq. (3.19), we find

$$\langle \phi \phi^* A^{a_1 \dots a_l} \rangle = \frac{x_{12}^{\Delta-2q-l}}{x_{13}^{\Delta-l} x_{23}^{\Delta-l}} Z^{a_1} \dots Z^{a_l} \quad (3.28)$$

$$\langle \phi \phi^* J^{aa_1 \dots a_l} \rangle = i(\Delta + l) \frac{x_{12}^{\Delta-2q-l}}{x_{13}^{\Delta-l} x_{23}^{\Delta-l}} Z^a Z^{a_1} \dots Z^{a_l} \quad (3.29)$$

$$\langle \phi \phi^* N^{a_2 \dots a_l} \rangle = i \frac{(\Delta - l - 2)(l + 1)}{2l} \frac{x_{12}^{\Delta-2q-l}}{x_{13}^{\Delta-l} x_{23}^{\Delta-l}} Z^2 Z^{a_2} \dots Z^{a_l} \quad (3.30)$$

$$\langle \phi \phi^* L^{aa_1 \dots a_l} \rangle = 2l \frac{x_{12}^{\Delta-2q-l}}{x_{13}^{\Delta-l} x_{23}^{\Delta-l}} Y^{aa_1} Z^{a_2} \dots Z^{a_l} \quad (3.31)$$

$$\begin{aligned} \langle \phi \phi^* D^{a_1 \dots a_l} \rangle = & \frac{x_{12}^{\Delta-2q-l}}{x_{13}^{\Delta-l} x_{23}^{\Delta-l}} \left(\frac{l}{2} Z^2 \frac{x_{21}^{a_1}}{x_{21}^2} Z^{a_2} \dots Z^{a_l} + \frac{l(l-1)}{2} \eta_{bc} Y^{a_1 b} Y^{a_2 c} Z^{a_3} \dots Z^{a_l} \right. \\ & \left. - \left(\frac{(\Delta + l)}{2} \frac{x_{32} \cdot x_{13}}{x_{13}^2 x_{32}^2} + \frac{(\Delta + l)(\Delta - l - 2)}{8} Z^2 \right) Z^{a_1} \dots Z^{a_l} \right), \end{aligned} \quad (3.32)$$

where

$$Z^a \equiv \frac{x_{31}^a}{x_{31}^2} - \frac{x_{32}^a}{x_{32}^2}, \quad Y^{ab} \equiv \frac{1}{x_{32}^2 x_{31}^2} \epsilon^{ab}{}_{cd} x_{31}^c x_{32}^d, \quad (3.33)$$

and we are implicitly projecting the right-hand side of each expression onto the appropriate Lorentz representation (symmetrizing and subtracting traces as necessary). Using $Z^2 = x_{12}^2/(x_{31}^2 x_{32}^2)$, we see that the correlators $\langle \phi \phi^* A \rangle, \langle \phi \phi^* J \rangle$ and $\langle \phi \phi^* N \rangle$ take the expected form for a 3-point function of conformal primary operators. Further, taking the appropriate derivatives of the above expressions and constructing the linear combinations corresponding to L_{prim} and D_{prim} , we obtain

$$\langle \phi \phi^* L_{\text{prim}}^{aa_1 \dots a_l} \rangle = 0 \quad (3.34)$$

as expected, and

$$\langle \phi \phi^* D_{\text{prim}}^{a_1 \dots a_l} \rangle = -\frac{\Delta(\Delta + l)(\Delta - l - 2)}{8(\Delta - 1)} \frac{x_{12}^{\Delta-2q-l}}{x_{13}^{\Delta-l} x_{23}^{\Delta-l}} Z^2 Z^{a_1} \dots Z^{a_l}. \quad (3.35)$$

Notice that three-point functions involving N and D_{prim} vanish when $\Delta = l + 2$, which is precisely what we expect for short multiplets that saturate the unitarity bound.

3.1.4 Conformal Primary Norms

Finally we must determine the normalization of the two-point functions $\langle JJ \rangle$, $\langle NN \rangle$, and $\langle D_{\text{prim}} D_{\text{prim}} \rangle$. One could do this either by expanding out the superconformally covariant expression for the two-point function of \mathcal{O} derived in [91] into its various components, or by using the explicit expressions for J, N , and D_{prim} in terms of Q, \bar{Q} , and P acting on A , and using the superconformal algebra to compute their norms in radial quantization. We here adopt the latter approach. We refer the reader to [94] for many examples of this type of computation.

To begin, we assume that the superconformal primary operator A is canonically normalized

$$\begin{aligned} \langle A^{b_1 \dots b_l} | A^{a_1 \dots a_l} \rangle &= \text{symmetrize}(\eta^{a_1 b_1} \dots \eta^{a_l b_l}) - \text{traces} \\ &= \frac{1}{l!} \sum_{\pi \in S_l} \eta^{a_1 b_{\pi(1)}} \dots \eta^{a_l b_{\pi(l)}} - \text{traces} \\ &\equiv \mathcal{I}_l^{a_1 \dots a_l; b_1 \dots b_l}, \end{aligned} \quad (3.36)$$

where we've defined $\mathcal{I}_l^{a_1 \dots a_l; b_1 \dots b_l}$ for future convenience, and $|A^{a_1 \dots a_l}\rangle = A^{a_1 \dots a_l}(0)|0\rangle$ is the state created by the operator $A^{a_1 \dots a_l}(0)$ in radial quantization.

Next we would like to determine the normalization of $B_{\text{prim}}^{aa_1 \dots a_l}$. Starting from Eqs. (3.20) and (3.26) and working through the algebra, we find that

$$\begin{aligned} \langle B_{\text{prim}}^{bb_1 \dots b_l} | B_{\text{prim}}^{aa_1 \dots a_l} \rangle &= 2 \left(\left(\Delta(\Delta + 1) - l^2 - \frac{l(l+1)}{\Delta-1} \right) \eta^{ba} \eta_{c_1}^{a_1} + l \left(2\Delta + 2l + 1 + \frac{l+1}{\Delta-1} \right) \eta^{ba_1} \eta_{c_1}^a \right. \\ &\quad \left. - l \left(2\Delta - 2l + 1 - \frac{l-1}{\Delta-1} \right) \eta_{c_1}^b \eta^{aa_1} \right) \langle A^{b_1 \dots b_l} | A^{c_1 a_2 \dots a_l} \rangle, \end{aligned} \quad (3.37)$$

from which we can extract the component normalizations

$$\langle J^{bb_1 \dots b_l} | J^{aa_1 \dots a_l} \rangle = 2(\Delta + l)(\Delta + l + 1) \mathcal{I}_{l+1}^{aa_1 \dots a_l; bb_1 \dots b_l}, \quad (3.38)$$

as well as

$$\langle N^{b_2 \dots b_l} | N^{a_2 \dots a_l} \rangle = \frac{2(l+1)^2}{l^2} (\Delta - l - 2)(\Delta - l - 1) \mathcal{I}_{l-1}^{a_2 \dots a_l; b_2 \dots b_l}, \quad (3.39)$$

where we have used the relation $\eta_{ab} \mathcal{I}_l^{aa_2 \dots a_l; bb_2 \dots b_l} = \frac{(l+1)^2}{l^2} \mathcal{I}_{l-1}^{a_2 \dots a_l; b_2 \dots b_l}$. Although we will not need it, for completeness we also have

$$\langle L_{\text{prim}}^{bb_1 \dots b_l} | L_{\text{prim}}^{aa_1 \dots a_l} \rangle = \frac{8l^2 \Delta(\Delta + l)(\Delta - l - 2)}{(l+1)^2(\Delta - 1)} \eta^{ab} \mathcal{I}_l^{a_1 \dots a_l; b_1 \dots b_l}, \quad (3.40)$$

where we are implicitly subtracting traces and the full symmetrization (in either the a, a_i or b, b_i indices) from the right hand side — that is, projecting onto the Lorentz representation corresponding to L .

Finally we must determine the normalization of $D_{\text{prim}}^{a_1 \dots a_l}$. In order to simplify the calculation, it will be helpful to write everything in terms of primary fields,

$$\begin{aligned} D_{\text{prim}}^{a_1 \dots a_l} = & -\frac{1}{64} \Xi_a B_{\text{prim}}^{aa_1 \dots a_l} - \frac{l(l+1) + (\Delta - 1)(\Delta + 1)}{16(\Delta - 1)^2} P^2 A^{a_1 \dots a_l} + \frac{l^2}{8(\Delta - 1)^2} P^{a_1} P_b A^{ba_2 \dots a_l} \\ & - \frac{3l}{16(\Delta - 1)} \epsilon^{a_1 b}{}_{cd} P_b B_{\text{prim}}^{cda_2 \dots a_l} \end{aligned} \quad (3.41)$$

so that

$$\begin{aligned} 64^2 \langle D_{\text{prim}}^{b_1 \dots b_l} | D_{\text{prim}}^{a_1 \dots a_l} \rangle = & \langle B_{\text{prim}}^{bb_1 \dots b_l} | (\Xi_b)^\dagger \Xi_a | B_{\text{prim}}^{aa_1 \dots a_l} \rangle - \frac{8l^2}{(\Delta - 1)^2} \langle B_{\text{prim}}^{bb_1 \dots b_l} | (\Xi_b)^\dagger P^{a_1} P_c | A^{ca_2 \dots a_l} \rangle \\ & + 4 \frac{l(l+1) + (\Delta - 1)(\Delta + 1)}{(\Delta - 1)^2} \langle B_{\text{prim}}^{bb_1 \dots b_l} | (\Xi_b)^\dagger P^2 | A^{a_1 \dots a_l} \rangle \\ & + \frac{12l}{(\Delta - 1)} \epsilon^{a_1 e}{}_{cd} \langle B_{\text{prim}}^{bb_1 \dots b_l} | (\Xi_b)^\dagger P_e | B_{\text{prim}}^{cda_2 \dots a_l} \rangle, \end{aligned} \quad (3.42)$$

where we have used that all terms of the form $\langle (\dots) K | D_{\text{prim}} \rangle$ vanish. Evaluating each of these terms using the superconformal algebra and putting everything together, we obtain the final result

$$\langle D_{\text{prim}}^{b_1 \dots b_l} | D_{\text{prim}}^{a_1 \dots a_l} \rangle = \frac{\Delta^2 (\Delta - l - 2)(\Delta - l - 1)(\Delta + l)(\Delta + l + 1)}{4(\Delta - 1)^2} \mathcal{I}_l^{a_1 \dots a_l; b_1 \dots b_l}. \quad (3.43)$$

3.1.5 $\mathcal{N} = 1$ Superconformal Blocks

To summarize the results in the previous subsections, we have found the three-point function coefficients

$$\begin{aligned}
\lambda_{\phi\phi^*A} &= 1 \\
\lambda_{\phi\phi^*J} &= i(\Delta + l) \\
\lambda_{\phi\phi^*N} &= i \frac{(\Delta - l - 2)(l + 1)}{2l} \\
\lambda_{\phi\phi^*D} &= - \frac{\Delta(\Delta + l)(\Delta - l - 2)}{8(\Delta - 1)}
\end{aligned} \tag{3.44}$$

and the norms

$$\begin{aligned}
\langle A|A \rangle &\sim 1 \\
\langle J|J \rangle &\sim 2(\Delta + l)(\Delta + l + 1) \\
\langle N|N \rangle &\sim \frac{2(l + 1)^2(\Delta - l - 2)(\Delta - l - 1)}{l^2} \\
\langle D|D \rangle &\sim \frac{\Delta^2(\Delta - l - 2)(\Delta - l - 1)(\Delta + l)(\Delta + l + 1)}{4(\Delta - 1)^2},
\end{aligned} \tag{3.45}$$

where “ \sim ” means multiplied by the appropriate canonically normalized tensor. Combining these results, we find the dimension Δ , spin l superconformal block given in Eq. (3.2), which we reproduce here for the reader’s convenience,

$$\begin{aligned}
\mathcal{G}_{\Delta,l} &= g_{\Delta,l} - \frac{(\Delta + l)}{2(\Delta + l + 1)} g_{\Delta+1,l+1} - \frac{(\Delta - l - 2)}{8(\Delta - l - 1)} g_{\Delta+1,l-1} \\
&\quad + \frac{(\Delta + l)(\Delta - l - 2)}{16(\Delta + l + 1)(\Delta - l - 1)} g_{\Delta+2,l}.
\end{aligned} \tag{3.46}$$

A few comments are in order. First, $l = 0$ is special, since in this case the N component does not exist. However, one can consistently take $g_{\Delta,-1} = 0$, and then the above equation correctly accounts for this situation. Second, in the case of superconformal primary operators that saturate the unitarity bound, $\Delta = l + 2$, the third and fourth terms vanish, which is precisely what we expect due to the fact that the N and D_{prim} components are not present in short multiplets. Finally, in

the case of the unit operator, with $\Delta = l = 0$, the second and fourth terms vanish due to the coefficient going to zero, and the third term vanishes because the conformal block goes to zero. Thus, we simply obtain that $\mathcal{G}_{0,0} = g_{0,0} = 1$.

Let us also note that Eq. (3.46) determines the superconformal blocks for four-point functions of all component fields in $\Phi(z_+)$, not just the lowest component $\phi(x)$. The reason is that there are unique superconformally-invariant extensions of the conformally-invariant cross-ratios u, v with the correct chirality properties to appear in a four-point function $\langle \Phi(z_{1+})\Phi^\dagger(z_{2-})\Phi(z_{3+})\Phi^\dagger(z_{4-}) \rangle$. They are given by [91]

$$\tilde{u} = \frac{x_{21}^2 x_{43}^2}{x_{23}^2 x_{41}^2}, \quad \tilde{v} = \frac{1}{2} \text{tr}(x_{21} x_{41}^{-1} x_{43} x_{23}^{-1}), \quad (3.47)$$

where the x 's in the trace should be thought of as bispinors, $(x)^{\dot{\alpha}\alpha} = x^a \bar{\sigma}_a^{\dot{\alpha}\alpha}$ and $(x^{-1})_{\alpha\dot{\alpha}} = -x_a \sigma_{\alpha\dot{\alpha}}^a / x^2$. Since \tilde{u} and \tilde{v} become u and v when we set $\theta_i = \bar{\theta}_i = 0$, we must have

$$\langle \Phi(z_{1+})\Phi^\dagger(z_{2-})\Phi(z_{3+})\Phi^\dagger(z_{4-}) \rangle = \frac{1}{x_{12}^{2d} x_{34}^{2d}} \sum_{\mathcal{O} \in \Phi \times \Phi^\dagger} \lambda_{\mathcal{O}}^2 \mathcal{G}_{\Delta,l}(\tilde{u}, \tilde{v}), \quad (3.48)$$

where $\mathcal{G}_{\Delta,l}$ is given by Eq. (3.46) above. One can now perform $\theta, \bar{\theta}$ expansions on both sides to derive the superconformal blocks for specific component fields.

Finally, let us mention that it may be possible to derive the superconformal blocks by mimicking the derivation of $g_{\Delta,l}$ in [95]. One would start with the expansion Eq. (3.48) and apply the quadratic casimir of the superconformal group acting on $\Phi(z_{1+})$ and $\Phi(z_{2-})$ to obtain a differential equation for $\mathcal{G}_{\Delta,l}$, which could then be solved.

3.1.6 Deriving $\mathcal{N} = 1$ Blocks From $\mathcal{N} = 2$ Blocks

In [78], Dolan and Osborn computed superconformal blocks for four-point functions of a particular kind of BPS operator in $\mathcal{N} = 2$ theories, using Ward identities special to higher supersymmetry. At the very least, we should be able to decompose their expression into $\mathcal{N} = 1$ superconformal blocks $\mathcal{G}_{\Delta,l}$. However, requiring that this is possible gives a strong consistency

condition on $\mathcal{G}_{\Delta,l}$ — so strong in fact that it determines $\mathcal{G}_{\Delta,l}$ completely! In this subsection, we will use this fact to give an alternate derivation of Eq. (3.46) that requires far less computation than in Sections 3.1.1-3.1.5, though it leverages important results from [78].

The operator φ^{ij} considered in [78] is a triplet under $SU(2)_R$, neutral under $U(1)_R$, and has scaling dimension 2 (here $i, j = 1, 2$ are $SU(2)_R$ indices). It satisfies the BPS conditions $Q_\alpha^{(i} \varphi^{jk)} = \epsilon^{l(i} \overline{Q}_{\dot{\alpha}l} \varphi^{jk)} = 0$, which imply that under the $\mathcal{N} = 1$ sub-algebra generated by Q_α^1 and $\overline{Q}_{\dot{\alpha}1}$, the operators φ^{11} , φ^{21} , and φ^{22} are anti-chiral, linear, and chiral respectively. The important fact for us is that $\varphi^{22} \equiv \phi$ is chiral, so $\langle \phi \phi^* \phi \phi^* \rangle$ can be decomposed into a sum of $\mathcal{G}_{\Delta,l}$'s. Note that the form of $\mathcal{G}_{\Delta,l}$ is independent of the dimension of ϕ . In particular, it is irrelevant for our purposes that ϕ is restricted to have dimension 2.

Any $\mathcal{N} = 2$ multiplet that can appear in the OPE $\varphi^{ij} \times \varphi^{kl}$ must be built from a primary of dimension Δ and definite integer spin l . We will denote such a multiplet by $(\Delta)_l^{\mathcal{N}=2}$. The “extra” supersymmetry generators Q^2, \overline{Q}_2 connect different $\mathcal{N} = 1$ multiplets within $(\Delta)_l^{\mathcal{N}=2}$ exactly analogously to the way Q and \overline{Q} connect different conformal multiplets within $(\Delta)_l^{\mathcal{N}=1}$, as discussed in Section 3.1.2. Thus, we have the decompositions

$$(\Delta)_l^{\mathcal{N}=2} = (\Delta)_l^{\mathcal{N}=1} \oplus (\Delta+1)_{l\pm 1}^{\mathcal{N}=1} \oplus (\Delta+2)_l^{\mathcal{N}=1} \quad (3.49)$$

$$(\Delta)_l^{\mathcal{N}=1} = (\Delta)_l^{\mathcal{N}=0} \oplus (\Delta+1)_{l\pm 1}^{\mathcal{N}=0} \oplus (\Delta+2)_l^{\mathcal{N}=0}, \quad (3.50)$$

where we have ignored multiplets which cannot appear in the OPE of two scalars. We can then write the ansatz

$$\mathcal{G}_{\Delta,l}^{\mathcal{N}=2} = \mathcal{G}_{\Delta,l} + N(\Delta, l) \mathcal{G}_{\Delta+1, l-1} + J(\Delta, l) \mathcal{G}_{\Delta+1, l+1} + D(\Delta, l) \mathcal{G}_{\Delta+2, l} \quad (3.51)$$

$$\mathcal{G}_{\Delta,l} = g_{\Delta,l} + n(\Delta, l) g_{\Delta+1, l-1} + j(\Delta, l) g_{\Delta+1, l+1} + d(\Delta, l) g_{\Delta+2, l}, \quad (3.52)$$

where N, J, D, n, j, d are functions we would like to determine. Note that j, n , and d must be rational functions of Δ and l . This is clear without any computation, simply from the viability of our first method for determining $\mathcal{G}_{\Delta,l}$ (Sections 3.1.1-3.1.5).

Using formulae from [78], we find that the $\mathcal{N} = 2$ superconformal block contributing to $\langle \phi \phi^* \phi \phi^* \rangle$ is given in terms of conformal blocks by

$$\begin{aligned}
\mathcal{G}_{\Delta,l}^{\mathcal{N}=2} = & g_{\Delta,l} - g_{\Delta+1,l+1} - \frac{1}{4}g_{\Delta+1,l-1} + \frac{1}{4}g_{\Delta+2,l} \\
& + \frac{(\Delta+l+2)^2}{4(\Delta+l+1)(\Delta+l+3)}g_{\Delta+2,l+2} - \frac{(\Delta+l+2)^2}{16(\Delta+l+1)(\Delta+l+3)}g_{\Delta+3,l+1} \\
& + \frac{(\Delta-l)^2}{64(\Delta-l-1)(\Delta-l+1)}g_{\Delta+2,l-2} - \frac{(\Delta-l)^2}{64(\Delta-l-1)(\Delta-l+1)}g_{\Delta+3,l-1} \\
& + \frac{(\Delta+l+2)^2(\Delta-l)^2}{256(\Delta+l+1)(\Delta+l+3)(\Delta-l-1)(\Delta-l+1)}g_{\Delta+4,l}.
\end{aligned} \tag{3.53}$$

Upon comparison with Eqs. (3.51) and (3.52), each coefficient in the above expression implies an equation relating N, J, D, n, j , and d . We will solve these equations by first determining j and n , and finally computing d in terms of them. To begin, the $g_{\Delta+1,l+1}$ and $g_{\Delta+2,l+2}$ terms in Eq. (3.53) imply

$$-1 = J(\Delta, l) + j(\Delta, l), \quad \text{and} \quad \frac{(\Delta+l+2)^2}{4(\Delta+l+1)(\Delta+l+3)} = J(\Delta, l)j(\Delta+1, l+1). \tag{3.54}$$

With some foresight, but without loss of generality, let us make the substitution

$$j(\Delta, l) = -\frac{(\Delta+l)}{2(\Delta+l+1)}(1 + \alpha(\Delta+l, \Delta-l)), \tag{3.55}$$

where $\alpha(x, \bar{x})$ is a rational function we must determine. Then Eqs. (3.54) imply the equation

$$\alpha(x, \bar{x}) = \frac{x+2}{x} \frac{\alpha(x+2, \bar{x})}{1 + \alpha(x+2, \bar{x})}, \tag{3.56}$$

and it's not difficult to show that any rational solution $\alpha(x, \bar{x})$ must vanish identically. Consequently, we obtain

$$j(\Delta, l) = -\frac{(\Delta+l)}{2(\Delta+l+1)}, \quad J(\Delta, l) = -\frac{(\Delta+l+2)}{2(\Delta+l+1)}. \tag{3.57}$$

A similar analysis using the $g_{\Delta+1,l-1}$ and $g_{\Delta+2,l-2}$ terms in Eq. (3.53) gives

$$n(\Delta, l) = -\frac{(\Delta-l-2)}{8(\Delta-l-1)}, \quad N(\Delta, l) = -\frac{(\Delta-l)}{8(\Delta-l-1)}. \tag{3.58}$$

Finally, let us solve for $d(\Delta, l)$. The $g_{\Delta+4, l}$ term in Eq. (3.53) determines $D(\Delta, l)$ in terms of $d(\Delta + 2, l)$. Plugging this in, along with our solutions for N, J, n , and j , the remaining terms in Eq. (3.53) imply equations with the following structure

$$g_{\Delta+2, l} : \quad d(\Delta, l) \sim d(\Delta + 2, l) \quad (3.59)$$

$$g_{\Delta+3, l+1} : \quad d(\Delta + 1, l + 1) \sim d(\Delta + 2, l) \quad (3.60)$$

$$g_{\Delta+3, l-1} : \quad d(\Delta + 1, l - 1) \sim d(\Delta + 2, l), \quad (3.61)$$

where “ \sim ” means “is algebraically related to.” Making the substitutions $\Delta \rightarrow \Delta - 1$ and $l \rightarrow l - 1$ in Eq. (3.60), we are left with three algebraic equations relating three “variables” $d(\Delta, l)$, $d(\Delta + 2, l)$, and $d(\Delta + 1, l - 1)$. Solving them gives

$$d(\Delta, l) = \frac{(\Delta + l)(\Delta - l - 2)}{16(\Delta + l + 1)(\Delta - l - 1)}, \quad D(\Delta, l) = \frac{(\Delta + l + 2)(\Delta - l)}{16(\Delta + l + 1)(\Delta - l - 1)}. \quad (3.62)$$

To summarize, we have re-derived Eq. (3.46),³ and also obtained the decomposition of $\mathcal{N} = 2$ conformal blocks into $\mathcal{N} = 1$ conformal blocks

$$\begin{aligned} \mathcal{G}_{\Delta, l}^{\mathcal{N}=2} &= \mathcal{G}_{\Delta, l} - \frac{(\Delta + l + 2)}{2(\Delta + l + 1)} \mathcal{G}_{\Delta+1, l+1} - \frac{(\Delta - l)}{8(\Delta - l - 1)} \mathcal{G}_{\Delta+1, l-1} \\ &\quad + \frac{(\Delta + l + 2)(\Delta - l)}{16(\Delta + l + 1)(\Delta - l - 1)} \mathcal{G}_{\Delta+2, l}. \end{aligned} \quad (3.63)$$

3.A Conventions

Our metric and spinor conventions are those of the $\eta_{ab} = \text{diag}(-1, +1, +1, +1)$ version of [96]. The Clifford relation is $\sigma_a \bar{\sigma}_b + \sigma_b \bar{\sigma}_a = -2\eta_{ab}$, so that one can convert between vectors and bispinors as $(x)_{\alpha\dot{\alpha}} = x_a \sigma_{\alpha\dot{\alpha}}^a$ and $x^a = -\frac{1}{2} \text{tr}(\bar{\sigma}^a x)$. These conventions agree with those of Wess and Bagger [88] and Osborn [91], with a single exception — the sign of σ^0 , which affects the coefficient

³It’s possible that similar arguments suffice to determine $\mathcal{N} = 2$ conformal blocks from $\mathcal{N} = 4$ conformal blocks. If this is the case, it’s fascinating that a maximally supersymmetric result, which can be derived using special properties of $\mathcal{N} = 4$ BPS multiplets, completely determines the corresponding results for lower supersymmetry.

of ϵ^{abcd} in products of σ 's and $\bar{\sigma}$'s. Specifically, we have

$$\sigma^a \bar{\sigma}^b \sigma^c = -\eta^{ab} \sigma^c + \eta^{ca} \sigma^b - \eta^{bc} \sigma^a - i\epsilon^{abcd} \sigma_d \quad (\text{this paper}) \quad (3.64)$$

$$\sigma^a \bar{\sigma}^b \sigma^c = -\eta^{ab} \sigma^c + \eta^{ca} \sigma^b - \eta^{bc} \sigma^a + i\epsilon^{abcd} \sigma_d \quad (\text{W\&B}) \quad (3.65)$$

To convert between these conventions, one simply flips the sign of ϵ^{abcd} wherever it appears.

For the $\mathcal{N} = 1$ superconformal algebra $\text{SU}(2, 2|1)$, we follow the conventions used in [97]; in particular we take bosonic generators to be *anti-hermitian* (that is, they differ from the usual definitions by a factor of i). This eliminates some factors of i from the commutation relations, somewhat simplifying the algebra in Section 3.1.

Let us arrange the superconformal generators according to their dimensions and spins as follows

dim(X)				
+1			P_a	
+1/2		Q_α		$\bar{Q}_{\dot{\alpha}}$
0	$M_{\alpha\beta}$		D, R	$M_{\dot{\alpha}\dot{\beta}}$
-1/2		S_α		$\bar{S}_{\dot{\alpha}}$
-1			$K_a,$	

(3.66)

where $M_{\alpha\beta} = (\sigma^{ba}\epsilon)_{\alpha\beta} M_{ab}$ and $M^{\dot{\alpha}\dot{\beta}} = (\bar{\sigma}^{ba}\epsilon)^{\dot{\alpha}\dot{\beta}} M_{ab}$ are self-dual and anti-self-dual rotation generators.

The dilatation operator and $U(1)_R$ generator act as

$$[D, X] = \dim(X)X \quad [R, X] = i r(X)X, \quad (3.67)$$

where X is any generator, $\dim(X)$ is given in the above table (3.66), and $r(X)$ is the R -charge of X , given by +1 for $X = S, \bar{Q}$, by -1 for $X = Q, \bar{S}$, and zero otherwise. The additional commutation

relations of the conformal sub-algebra are given by

$$\begin{aligned}
[M_{ab}, P_c] &= P_a \eta_{bc} - P_b \eta_{ac}, & [M_{ab}, K_c] &= K_a \eta_{bc} - K_b \eta_{ac} \\
[M_{ab}, M_{cd}] &= \eta_{bc} M_{ad} - \eta_{ac} M_{bd} - \eta_{bd} M_{ac} + \eta_{ad} M_{bc} \\
[K_a, P_b] &= 2\eta_{ab} D - 2M_{ab}.
\end{aligned} \tag{3.68}$$

Rotation generators act on spinors as

$$[M_{ab}, X_\alpha] = (\sigma_{ab})_\alpha{}^\beta X_\beta, \quad [M_{ab}, \bar{X}^{\dot{\alpha}}] = (\bar{\sigma}_{ab})^{\dot{\alpha}}{}_{\dot{\beta}} \bar{X}^{\dot{\beta}}, \tag{3.69}$$

where $X_\alpha = S, Q$ and $\bar{X} = \bar{Q}, \bar{S}$. Finally, the remaining non-vanishing commutation relations involving fermionic generators are

$$\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = -2i\sigma_{\alpha\dot{\alpha}}^a P_a, \quad \{S_\alpha, \bar{S}_{\dot{\alpha}}\} = +2i\sigma_{\alpha\dot{\alpha}}^a K_a \tag{3.70}$$

$$[K_a, Q_\alpha] = i\sigma_{a\alpha\dot{\beta}} \bar{S}^{\dot{\beta}}, \quad [S_\alpha, P_a] = i\sigma_{a\alpha\dot{\beta}} \bar{Q}^{\dot{\beta}} \tag{3.71}$$

$$[K_a, \bar{Q}^{\dot{\alpha}}] = i\bar{\sigma}_a^{\dot{\alpha}\beta} S_\beta, \quad [\bar{S}^{\dot{\alpha}}, P_a] = i\bar{\sigma}_a^{\dot{\alpha}\beta} Q_\beta, \tag{3.72}$$

$$\{S_\alpha, Q_\beta\} = 2D\epsilon_{\alpha\beta} - 2M_{\alpha\beta} - 3iR\epsilon_{\alpha\beta}, \tag{3.73}$$

$$\{\bar{S}^{\dot{\alpha}}, \bar{Q}^{\dot{\beta}}\} = 2D\epsilon^{\dot{\alpha}\dot{\beta}} - 2M^{\dot{\alpha}\dot{\beta}} + 3iR\epsilon^{\dot{\alpha}\dot{\beta}}. \tag{3.74}$$

The relation between our conventions for the super-Poincaré subalgebra, and those of Wess and Bagger is summarized by equating supergroup elements at each point $(x, \theta, \bar{\theta})$ in superspace

$$\left[e^{x \cdot P + \theta Q + \bar{\theta} \bar{Q}} \right]_{\text{this paper}} = \left[e^{i(-x \cdot P + \theta Q + \bar{\theta} \bar{Q})} \right]_{\text{W\&B}}. \tag{3.75}$$

In particular, component expansions of our superfields $\mathcal{O}(x, \theta, \bar{\theta}) = e^{x \cdot P + \theta Q + \bar{\theta} \bar{Q}} \mathcal{O}(0)$ are the same as component expansions in Wess and Bagger, with the only difference being an overall factor of i or $-i$ in the action of super-Poincaré generators.

Chapter 4

Carving Out the Space of 4D CFTs

This chapter is a lightly-edited version of

D. Poland, D. Simmons-Duffin and A. Vichi, “Carving Out the Space of 4D CFTs,”
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4.1 Introduction

In recent years it has been realized that the restrictions imposed by conformal symmetry are not very well understood. While constraints on the form of simple correlation functions (e.g., [98, 99]) and unitarity restrictions on operator dimensions [100, 101] were worked out long ago, it was pointed out in [54] that crossing symmetry of four-point functions combined with the constraints of unitarity imply additional bounds on operator dimensions that must be satisfied in any consistent CFT. These bounds were soon strengthened [55] and extended to bounds on scalar operator product expansion (OPE) coefficients [56]. In [57] the bounds were also extended to $\mathcal{N} = 1$ superconformal field theories (SCFTs); bounds on central charges in general CFTs and SCFTs were also explored in [57] and [58]. In addition, progress on incorporating global symmetries into the program (important for both phenomenological applications and to have a more direct comparison

with known theories) was made in [59], and improved bounds (both for general CFTs with global symmetries and for SCFTs) were presented in [60].

The methods used in [54–60] to obtain bounds involve applying linear functionals to CFT crossing relations, which in practice means taking linear combinations of derivatives of the crossing relations evaluated at a particular point. By searching for linear functionals that are positive when acting on the contributions of all possible primary operators in the spectrum other than the unit operator, one can obtain bounds on OPE coefficients (and sometimes operator dimensions). However, to implement this positivity condition, the authors of [54–60] introduced a finely-spaced discretization of the set of possible operator dimensions, making the resulting *linear programming* problem numerically difficult and limiting how far the idea could be pushed. This numerical limitation was particularly apparent when considering systems of crossing relations that occur in theories with global symmetries, where the bounds obtained so far still seem to be quite far from their optimal values.

In the present paper we will present an alternate approach that completely avoids this discretization of dimensions. We will use the fact that linear combinations of derivatives of conformal blocks can be arbitrarily-well approximated by ratios of polynomials in the operator dimensions, which allows us to convert the problem of obtaining bounds into a *semidefinite programming* problem that is numerically much more efficient. This then allows us to obtain much stronger bounds on CFTs and SCFTs, particularly in the presence of global symmetries.

More concretely, for general CFTs we will consider four-point functions of scalar operators ϕ , as well as collections of operators ϕ_i transforming as fundamentals under $\text{SO}(N)$ or $\text{SU}(N)$ global symmetries. For theories with $\mathcal{N} = 1$ supersymmetry we will focus on the case of chiral superconformal primary operators Φ , as well as on collections of chiral operators Φ_i transforming as $\text{SU}(N)$ fundamentals. We start by reviewing the relevant crossing relations and representation theory in section 4.2. There we will also introduce our new method to obtain bounds on operator

dimensions and OPE coefficients based on semidefinite programming.

In section 4.3 we use this method to derive general bounds on operator dimensions. In the case of general CFTs with $\text{SO}(N)$ global symmetries, we will place upper bounds on the dimension of the lowest-dimension $\text{SO}(N)$ -singlet operator appearing in the $\phi_i \times \phi_j$ OPE. This greatly improves upon the bounds in the presence of global symmetries previously presented in [59, 60]. We also place similar bounds on the lowest-dimension $\text{SO}(N)$ symmetric tensor $\phi_{(i}\phi_{j)}$. In the case of $\text{SU}(N)$ global symmetries we can additionally place bounds on $\text{SU}(N)$ -singlet or $\text{SU}(N)$ -adjoint operators appearing in the $\phi_i \times \phi^{\bar{j}}$ OPE. Somewhat surprisingly, we find that $\text{SU}(N)$ -singlet bounds turn out to be identical to $\text{SO}(2N)$ -singlet bounds using the present method.

The special case of an $\text{SO}(4)$ or $\text{SU}(2)$ global symmetry is relevant for the scenario of conformal technicolor [8], with or without custodial symmetry. In this scenario one would like the dimension of the Higgs operator H to be somewhat close to 1, while the dimension of $H^\dagger H$ should be close to or greater than 4. On the other hand, the bounds in this paper show that requiring $\dim(H^\dagger H) \geq 4$ forces one to have at least $\dim(H) \gtrsim 1.52$, excluding flavor-generic versions of this scenario and placing significant constraints on models where Yukawa-like suppressions are generated in four-fermion operators.

In $\mathcal{N} = 1$ superconformal theories we also place bounds on the lowest-dimension scalar superconformal primary appearing in the $\Phi \times \Phi^\dagger$ OPE, where Φ is a chiral operator. This greatly strengthens the bounds presented in [57, 60]. In fact, we will see that the bound appears to asymptote to the line $\dim(\Phi^\dagger \Phi) \leq 2 \dim(\Phi)$ near $\dim(\Phi) \sim 1$, essentially excluding the possibility of ‘positive anomalous dimensions’ (as recently discussed in [102]) in this region. This also implies that the solution to the $\mu/B\mu$ -problem proposed in [32, 33] cannot easily work near $\dim(\Phi) \sim 1$.

In section 4.4 we explore bounds on OPE coefficients. First we strengthen the upper bounds presented in [56] on the sizes of OPE coefficients of scalars \mathcal{O} appearing in the $\phi \times \phi$ OPE in non-supersymmetric theories. Then, as a new application of these methods in superconformal

theories, we place both upper and *lower* bounds the OPE coefficient of the chiral Φ^2 operator which always appears in the $\Phi \times \Phi$ OPE. In this case, lower bounds are possible because unitarity requires that there is a gap in the spectrum of dimensions, so no other nearby operators can mimic the effects of the Φ^2 operator in the conformal block decomposition. We similarly place upper and lower bounds on the OPE coefficients of the other higher-spin protected operators that can appear in the $\Phi \times \Phi$ OPE. These bounds have interesting implications for Banks-Zaks theories or CFTs with weakly-coupled AdS_5 duals, where they can be checked in perturbation theory.

Next, in section 4.5 we place lower bounds on the central charge c , which appears as the coefficient in the two-point function of the stress tensor: $\langle TT \rangle \propto c$. These bounds strengthen and expand upon those previously explored in [57, 58, 60]. In theories with operators of dimension d transforming as fundamentals under $\text{SO}(N)$ or $\text{SU}(N)$ global symmetries, we find that the bounds scale linearly with N near $d \sim 1$, consistent with our intuition from free CFTs. We explore these bounds on c in both general CFTs and $\mathcal{N} = 1$ SCFTs. In the latter case one can calculate c using 't Hooft anomaly matching in many known SCFTs, and our bounds are satisfied in all such examples that we have checked.

In section 4.6 we place similar bounds on the coefficient κ appearing in the two-point function of a global symmetry current: $\langle J^A J^B \rangle \propto \kappa \text{Tr}(T^A T^B)$. Here we extend the previous results of [57] to include the full information about global symmetries. In the case of scalar operators transforming as fundamentals of $\text{SO}(N)$, we place lower bounds on $\kappa_{\text{SO}(N)}$. In the case of $\text{SU}(N)$ global symmetries, one can either bound the OPE coefficient appearing in front of the $\text{SU}(N)$ (adjoint) current or the coefficient in front of an $\text{SU}(N)$ -singlet current corresponding to a different global symmetry. In the latter case, the bounds again scale linearly with N near $d \sim 1$ in accordance with our intuition from free CFTs. We also compute similar bounds in $\mathcal{N} = 1$ SCFTs where κ can be computed using 't Hooft anomaly matching, and present a comparison of our results with supersymmetric QCD in the conformal window [103]. We conclude in section 4.7.

4.2 Bounds from Crossing Relations

4.2.1 CFT Review

Let us begin by reviewing some basic aspects of conformal field theories that will be important for our discussion. The conformal algebra contains, in addition to Poincaré generators, a dilatation generator D and special conformal generators K_μ . Operators in a CFT can be classified into primaries \mathcal{O}^I satisfying $K_\mu \mathcal{O}^I(0) = 0$, and their descendants $P^\mu \cdots P^\nu \mathcal{O}^I(0)$.¹ Here, I denotes possible Lorentz indices. We will be primarily concerned with spin- ℓ operators which transform as traceless symmetric tensors of the Lorentz group, $\mathcal{O}^I = \mathcal{O}^{\mu_1 \cdots \mu_\ell}$.

Correlation functions of a conformal field theory on \mathbb{R}^n are completely determined by some simple discrete data: the spectrum of operator dimensions and spins, and the coefficients appearing in the operator product expansion (OPE). Knowledge of the spectrum is sufficient to determine all two-point functions. For primary operators \mathcal{O}_i^I and \mathcal{O}_j^J with equal dimensions and spins $\{\Delta, \ell\}$, we have

$$\langle \mathcal{O}_i^I(x_1) \mathcal{O}_j^J(x_2) \rangle \propto \frac{w^{IJ}(x_{12})}{x_{12}^{2\Delta}}, \quad (4.1)$$

where $w^{IJ}(x)$ is a tensor whose form is fixed by conformal symmetry (e.g., for spin-1 operators $w^{\mu\nu}(x) = \eta^{\mu\nu} - \frac{2x^\mu x^\nu}{x^2}$). When the dimensions and spins are not equal, the two-point function must vanish. In addition, unitarity constrains Δ to satisfy [100, 101]

$$\begin{aligned} \Delta &\geq 1 & (\ell = 0), \\ \Delta &\geq \ell + 2 & (\ell \geq 1). \end{aligned} \quad (4.2)$$

These bounds can sometimes be strengthened if the conformal algebra is enhanced, as in superconformal theories. We will see some examples of this shortly.

¹We leave the adjoint action of charges on operators implicit, i.e. $K_\mu \mathcal{O} \equiv [K_\mu, \mathcal{O}]$.

Let us choose an orthonormal basis of primaries \mathcal{O}_i , so that the constant of proportionality in Eq. (4.1) is δ_{ij} . Having done so, the remaining n -point functions of the theory are determined by coefficients in the operator product expansion. For real scalars ϕ_1 and ϕ_2 , this takes the form [104]

$$\phi_1(x)\phi_2(0) = \sum_{\mathcal{O} \in \phi_1 \times \phi_2} \lambda_{\phi_1 \phi_2 \mathcal{O}} C_I(x, P) \mathcal{O}^I(0), \quad (4.3)$$

where $\lambda_{\phi_1 \phi_2 \mathcal{O}}$ are constants that must be real in a unitary theory. The notation $\mathcal{O} \in \phi_1 \times \phi_2$ indicates that \mathcal{O} is a primary operator in the OPE of ϕ_1 and ϕ_2 . We have grouped together each primary \mathcal{O} and its descendants $P\mathcal{O}, P^2\mathcal{O}, \dots$ into a single term using the operator $C_I(x, P)$ (which depends on the dimensions and spins of ϕ_1, ϕ_2 , and \mathcal{O} , though we are suppressing that dependence for brevity). One can show that the form of $C_I(x, P)$ is completely fixed by conformal symmetry. For instance, applying special conformal generators K_μ to both sides of Eq. (4.3) gives a recursion relation for the terms in C_I which can be solved order-by-order. When $\phi_1 = \phi_2 = \phi$, Bose symmetry dictates that only even-spin operators may enter the OPE (4.3).

In general field theories, the OPE is an asymptotic expansion, valid only at short distances. However in a CFT, because of the absence of scales, the OPE is an exact equality that can be used to simplify products of operators with arbitrary separation inside correlation functions.² A key example for us is a four-point function of a scalar operator ϕ of dimension d , which can be evaluated as follows:

$$\begin{aligned} \langle \overline{\phi(x_1)\phi(x_2)} \overline{\phi(x_3)\phi(x_4)} \rangle &= \sum_{\substack{\mathcal{O} \in \phi \times \phi \\ \mathcal{O}' \in \phi \times \phi}} \lambda_{\mathcal{O}} \lambda_{\mathcal{O}'} C_I(x_{12}, \partial_2) C_J(x_{34}, \partial_4) \langle \mathcal{O}^I(x_2) \mathcal{O}'^J(x_4) \rangle \\ &= \sum_{\mathcal{O} \in \phi \times \phi} \lambda_{\mathcal{O}}^2 \frac{1}{x_{12}^{2d} x_{34}^{2d}} g_{\Delta, \ell}(u, v), \end{aligned} \quad (4.4)$$

$$g_{\Delta, \ell}(u, v) \equiv x_{12}^{2d} x_{34}^{2d} C_I(x_{12}, \partial_2) C_J(x_{34}, \partial_4) \frac{w^{IJ}(x_{24})}{x_{24}^{\Delta}}, \quad (4.5)$$

where we have inserted Eq. (4.3) twice and used Eq. (4.1) together with orthonormality of the \mathcal{O} 's.

²This is true provided there are no other operators ‘nearby’ in a sense that can be made precise.

Here, Δ and ℓ are the dimension and spin of \mathcal{O} , $u \equiv \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}$ and $v \equiv \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$ are conformal cross-ratios, and the functions $g_{\Delta,\ell}(u, v)$ are called *conformal blocks*. Since conformal symmetry completely fixes C_I and w^{IJ} , it also determines $g_{\Delta,\ell}$. An exact expression in four dimensions, computed by Dolan and Osborn [95, 105], is given by

$$g_{\Delta,\ell}(u, v) = \frac{z\bar{z}}{z - \bar{z}} (k_{\Delta+\ell}(z) k_{\Delta-\ell-2}(\bar{z}) - (z \leftrightarrow \bar{z})), \quad (4.6)$$

$$k_\beta(x) \equiv x^{\beta/2} {}_2F_1(\beta/2, \beta/2, \beta, x), \quad (4.7)$$

where $u = z\bar{z}$ and $v = (1 - z)(1 - \bar{z})$.³ The unit operator is an important special case, with $g_{0,0}(u, v) = 1$.

4.2.2 Crossing Relations for Singlets, $\text{SO}(N)$, and $\text{SU}(N)$

While a set of dimensions, spins, and OPE coefficients is enough to compute any correlation function, this data must satisfy additional consistency relations in a sensible CFT. To simplify $\langle \phi\phi\phi\phi \rangle$ using the OPE, we had to choose some way of pairing up the operators, and this choice necessarily broke manifest permutation symmetry among the $\phi(x_i)$'s. Nevertheless it should be the case that the end result remains permutation-symmetric, a requirement known as crossing symmetry. As an example, switching $x_1 \leftrightarrow x_3$ in the conformal block expansion Eq. (4.5) leads to the crossing relation

$$\sum_{\mathcal{O} \in \phi \times \phi} \lambda_{\mathcal{O}}^2 g_{\Delta,\ell}(u, v) = \left(\frac{u}{v}\right)^d \sum_{\mathcal{O} \in \phi \times \phi} \lambda_{\mathcal{O}}^2 g_{\Delta,\ell}(v, u). \quad (4.8)$$

Meanwhile, switching $x_1 \leftrightarrow x_2$ reproduces the statement that only even-spin primaries appear in $\phi \times \phi$. Other permutations give no new information in this case.⁴

³Our convention for conformal blocks here differs by a factor of $(-2)^\ell$ from the one used in [95, 105].

⁴Note that crossing symmetry of all four-point functions is equivalent to associativity of the OPE, which is enough to guarantee that higher n -point functions are crossing-symmetric as well.

Recall that the $\lambda_{\mathcal{O}}$ are real by unitarity, which means that the coefficients $\lambda_{\mathcal{O}}^2$ are non-negative. This is a source of tension in Eq. (4.8), which can be expressed most clearly by rewriting our crossing relation as a ‘sum rule’ with positive coefficients,

$$F_{0,0}(u,v) + \sum_{\mathcal{O} \in \phi \times \phi} \lambda_{\mathcal{O}}^2 F_{\Delta,\ell}(u,v) = 0, \quad (4.9)$$

where

$$F_{\Delta,\ell}(u,v) \equiv \frac{v^d g_{\Delta,\ell}(u,v) - u^d g_{\Delta,\ell}(v,u)}{u^d - v^d}, \quad (4.10)$$

$$F_{0,0}(u,v) = -1, \quad (4.11)$$

and we are suppressing the d -dependence of $F_{\Delta,\ell}$ for brevity. Note that we have isolated the term corresponding to the unit operator, whose OPE coefficient is fixed by the fact that ϕ has a canonically normalized two-point function. The unit operator contributes to Eq. (4.9) in its own particular way. Requiring that this contribution be cancelled by $F_{\Delta,\ell}$ ’s with positive coefficients leads to nontrivial constraints on the allowed Δ, ℓ appearing in $\phi \times \phi$. For some explicit examples and many details about the structure of the sum rule, see [54, 55]. In sections 4.2.4 and 4.2.5, we will explain our improved method for extracting bounds on CFT data from Eq. (4.9). For now, let us present some generalizations of the sum rule for other kinds of operators.

SO(N) Crossing Relations

An analysis of crossing relations in theories with SO(N) and SU(N) global symmetries was performed in [59], and improved bounds for SO(N) were presented in [60]. We will make extensive use of these results, so let us review them here.

Consider a real scalar primary ϕ_i transforming in the fundamental representation of an SO(N) global symmetry group. A complex scalar is a special case with symmetry group SO(2) \cong U(1). Operators in $\phi_i \times \phi_j$ can be organized into singlets S , symmetric tensors T , and antisymmetric

tensors A of $\text{SO}(N)$. Schematically,

$$\phi_i \times \phi_j \sim \sum_{S^+} \delta_{ij} \mathcal{O} + \sum_{T^+} \mathcal{O}_{(ij)} + \sum_{A^-} \mathcal{O}_{[ij]}. \quad (4.12)$$

The notation S^\pm, T^\pm, A^\pm indicates that the sum is restricted to even-spin (+) or odd-spin (−) primaries in $\phi_i \times \phi_j$ with the given representation, as dictated by Bose symmetry. Keeping track of the $\text{SO}(N)$ indices, each representation contributes differently to the conformal block decomposition of a four-point function,

$$\begin{aligned} x_{12}^{2d} x_{34}^{2d} \langle \phi_i(x_1) \phi_j(x_2) \phi_k(x_3) \phi_l(x_4) \rangle &= \sum_{S^+} \lambda_{\mathcal{O}}^2 (\delta_{ij} \delta_{kl}) g_{\Delta, \ell}(u, v) \\ &+ \sum_{T^+} \lambda_{\mathcal{O}}^2 \left(\delta_{il} \delta_{jk} + \delta_{ik} \delta_{jl} - \frac{2}{N} \delta_{ij} \delta_{kl} \right) g_{\Delta, \ell}(u, v) \\ &+ \sum_{A^-} \lambda_{\mathcal{O}}^2 (\delta_{il} \delta_{jk} - \delta_{ik} \delta_{jl}) g_{\Delta, \ell}(u, v). \end{aligned} \quad (4.13)$$

If we recompute this four-point function using a different operator pairing, each primary contributes again, but with the conformal cross-ratios u and v switched, and the tensor structures $\delta_{ij} \delta_{kl}$, $\delta_{ik} \delta_{jl}$, $\delta_{il} \delta_{jk}$ permuted. Picking out the coefficients of each tensor structure then leads to three sum rules, which we can write in vectorial form

$$\sum_{S^+} \lambda_{\mathcal{O}}^2 \begin{pmatrix} 0 \\ F_{\Delta, \ell} \\ H_{\Delta, \ell} \end{pmatrix} + \sum_{T^+} \lambda_{\mathcal{O}}^2 \begin{pmatrix} F_{\Delta, \ell} \\ (1 - \frac{2}{N}) F_{\Delta, \ell} \\ -(1 + \frac{2}{N}) H_{\Delta, \ell} \end{pmatrix} + \sum_{A^-} \lambda_{\mathcal{O}}^2 \begin{pmatrix} -F_{\Delta, \ell} \\ F_{\Delta, \ell} \\ -H_{\Delta, \ell} \end{pmatrix} = 0. \quad (4.14)$$

Here $H_{\Delta, \ell}(u, v)$ is a symmetrized version of $F_{\Delta, \ell}(u, v)$,

$$H_{\Delta, \ell}(u, v) \equiv \frac{v^d g_{\Delta, \ell}(u, v) + u^d g_{\Delta, \ell}(v, u)}{u^d + v^d}, \quad (4.15)$$

$$H_{0,0}(u, v) = 1. \quad (4.16)$$

For brevity, we have not isolated the unit operator in Eq. (4.14); it is included with the even-spin singlets S^+ .

The case of $\text{SO}(4)$ is special, since one can additionally decompose antisymmetric tensors into self-dual and anti-self-dual parts A_{\pm} . Let us quickly summarize the consequences, though they will turn out to be irrelevant for this work. A new tensor structure can now appear in $\langle \phi_i \phi_j \phi_k \phi_l \rangle$, namely ϵ_{ijkl} . In Eq. (4.13) we must replace

$$\sum_{A^-} \lambda_{\mathcal{O}}^2 (\delta_{il} \delta_{jk} - \delta_{ik} \delta_{jl}) g_{\Delta, \ell}(u, v) \rightarrow \sum_{A_{\pm}^-} \lambda_{\mathcal{O}}^2 (\delta_{il} \delta_{jk} - \delta_{ik} \delta_{jl} \pm \epsilon_{ijkl}) g_{\Delta, \ell}(u, v). \quad (4.17)$$

Since ϵ_{ijkl} maps to itself under permutations, the sum rule Eq. (4.14) is unaffected. We must simply supplement it with

$$\sum_{A_+^-} \lambda_{\mathcal{O}}^2 F_{\Delta, \ell} - \sum_{A_-^-} \lambda_{\mathcal{O}}^2 F_{\Delta, \ell} = 0. \quad (4.18)$$

Before we proceed, it is worth mentioning that all three of the sum rules given in Eq. (4.14) can be derived from a single ‘master’ crossing relation

$$\sum_{S^+} g_{\Delta, \ell}(u, v) - \frac{2}{N} \sum_{T^+} g_{\Delta, \ell}(u, v) = \left(\frac{u}{v}\right)^d \left(\sum_{T^+} g_{\Delta, \ell}(v, u) + \sum_{A^-} g_{\Delta, \ell}(v, u) \right). \quad (4.19)$$

Adding Eq. (4.19) to itself with $u \leftrightarrow v$ gives the second row of Eq. (4.14) and subtracting it from itself with $u \leftrightarrow v$ gives the third row. To obtain the first row, we must make repeated use of the identity $g(u, v) = (-1)^l g(u/v, 1/v)$:

$$\begin{aligned} \sum_{T^+} g_{\Delta, \ell}(u, v) - \sum_{A^-} g_{\Delta, \ell}(u, v) &= \sum_{T^+} g_{\Delta, \ell}(u/v, 1/v) + \sum_{A^-} g_{\Delta, \ell}(u/v, 1/v) \\ &= u^d \left(\sum_{S^+} g_{\Delta, \ell}(1/v, u/v) - \frac{2}{N} \sum_{T^+} g_{\Delta, \ell}(1/v, u/v) \right) \\ &= u^d \left(\sum_{S^+} g_{\Delta, \ell}(1/u, v/u) - \frac{2}{N} \sum_{T^+} g_{\Delta, \ell}(1/u, v/u) \right) \\ &= \left(\frac{u}{v}\right)^d \left(\sum_{T^+} g_{\Delta, \ell}(v/u, 1/u) + \sum_{A^-} g_{\Delta, \ell}(v/u, 1/u) \right) \\ &= \left(\frac{u}{v}\right)^d \left(\sum_{T^+} g_{\Delta, \ell}(v, u) - \sum_{A^-} g_{\Delta, \ell}(v, u) \right). \end{aligned} \quad (4.20)$$

This implies in particular that the first sum rule is not independent from the other two. However, in practice we find it useful to retain all three sum rules, since we will keep only a finite number of terms in their Taylor expansions around a single point in (u, v) -space. Since the derivation Eq. (4.20) requires transformation between different (u, v) points, the exact equivalence between the third sum rule and the other two is only visible with an infinite number of terms in the Taylor expansion. However, it will be important to clarify the meaning of this ‘master’ sum rule (and its generalization to other symmetries) in future studies.

SU(N) Crossing Relations

Let us now consider a complex scalar ϕ_i transforming in the fundamental representation of an SU(N) global symmetry. For this paper, we will only analyze four-point functions $\langle \phi_i \phi^{\bar{j}\dagger} \phi_k \phi^{\bar{l}\dagger} \rangle$ that would be invariant under an additional $U(1)$ acting on ϕ . Note that this is not tantamount to assuming such a $U(1)$ exists — rather, we are restricting our attention to a subset of CFT correlators. The various channels for decomposing our four-point function now involve two different kinds of OPEs. Firstly,

$$\phi_i \times \phi^{\bar{j}\dagger} \sim \sum_{S^\pm} \delta_i^{\bar{j}} \mathcal{O} + \sum_{\text{Ad}^\pm} \mathcal{O}_i^{\bar{j}} \quad (4.21)$$

which can contain SU(N) singlets and adjoints of any spin. We also have

$$\phi_i \times \phi_j \sim \sum_{T^+} \mathcal{O}_{(ij)} + \sum_{A^-} \mathcal{O}_{[ij]} \quad (4.22)$$

containing symmetric and antisymmetric tensors with even and odd spins, respectively, and its complex conjugate $\phi^{\bar{i}\dagger} \times \phi^{\bar{j}\dagger}$ containing the conjugate operators in dual representations. Extracting the coefficients of different tensor structures in all possible ways of evaluating $\langle \phi_i \phi^{\bar{j}\dagger} \phi_k \phi^{\bar{l}\dagger} \rangle$ leads to the six-fold sum rule

$$\sum_{S^\pm} \lambda_{\mathcal{O}}^2 V_{\Delta, \ell}^{S^\pm} + \sum_{\text{Ad}^\pm} \lambda_{\mathcal{O}}^2 V_{\Delta, \ell}^{\text{Ad}^\pm} + \sum_{T^+} \lambda_{\mathcal{O}}^2 V_{\Delta, \ell}^{T^+} + \sum_{A^-} \lambda_{\mathcal{O}}^2 V_{\Delta, \ell}^{A^-} = 0, \quad (4.23)$$

where

$$V^{S^\pm} = \begin{pmatrix} F \\ H \\ (-)^\ell F \\ (-)^\ell H \\ 0 \\ 0 \end{pmatrix}, \quad V^{\text{Ad}^\pm} = \begin{pmatrix} (1 - \frac{1}{N})F \\ -(1 + \frac{1}{N})H \\ -(-)^\ell \frac{1}{N}F \\ -(-)^\ell \frac{1}{N}H \\ (-)^\ell F \\ (-)^\ell H \end{pmatrix}, \quad V^{T^+} = \begin{pmatrix} 0 \\ 0 \\ F \\ -H \\ F \\ -H \end{pmatrix}, \quad V^{A^-} = \begin{pmatrix} 0 \\ 0 \\ F \\ -H \\ -F \\ H \end{pmatrix}. \quad (4.24)$$

Once again, the unit operator is included among even-spin singlets S^+ .

4.2.3 Crossing Relations in Superconformal Theories

The 4D $\mathcal{N} = 1$ superconformal algebra extends the conformal algebra to include supersymmetry generators $Q_\alpha, \bar{Q}_{\dot{\alpha}}$, superconformal generators $S_\alpha, \bar{S}_{\dot{\alpha}}$, and a $U(1)$ R -charge generator. SCFT operators admit a more refined classification into *superconformal primaries* satisfying $S\mathcal{O}(0) = \bar{S}\mathcal{O}(0) = 0$, with their superconformal descendants obtained by acting with any combination of Q, \bar{Q} , and P . It's easy to see using $\{S, \bar{S}\} \sim K$ that a superconformal primary is also a conformal primary. But the converse is not necessarily true. A multiplet built from a single superconformal primary generally contains several (though finitely many) conformal primaries whose dimensions, spins, and OPE coefficients are related by supersymmetry.

A principle example for this work is a chiral superconformal primary scalar Φ of dimension d , which satisfies $S\Phi(0) = \bar{S}\Phi(0) = \bar{Q}\Phi(0)$. Unitarity implies that its dimension is proportional to its R -charge, $d = \frac{3}{2}R_\Phi$. Below we will review the structure of the OPEs needed to decompose four-point functions of Φ and Φ^\dagger into conformal blocks. We also refer the reader to [57, 60, 79] for additional discussions of these OPEs.

First, superconformal primaries \mathcal{O} appearing in $\Phi \times \Phi^\dagger$ are restricted to have vanishing R -charge and a dimension satisfying the unitarity bound $\Delta \geq 2 + \ell$, where $\ell \geq 0$ is the spin of \mathcal{O} . Each

superconformal primary generically comes with three superconformal descendants of definite spin which are also primaries under the conformal subalgebra. (When the unitarity bound is saturated, $\Delta = 2 + \ell$, two of these descendants vanish and the multiplet is shortened.) Schematically, the OPE takes the form

$$\Phi \times \Phi^\dagger \sim \sum_{\mathcal{O} \in \Phi \times \Phi^\dagger} \left[\mathcal{O} + (Q\bar{Q}\mathcal{O})_{\ell-1} + (Q\bar{Q}\mathcal{O})_{\ell+1} + Q^2\bar{Q}^2\mathcal{O} \right], \quad (4.25)$$

where $\mathcal{O} \in \Phi \times \Phi^\dagger$ denotes that the sum is over *superconformal* primaries in $\Phi \times \Phi^\dagger$, and the subscript on $Q\bar{Q}\mathcal{O}$ indicates the spin. We are being somewhat sketchy in our notation; the exact form of these conformal primaries depends on Δ and ℓ and is given in [57]. Superconformal symmetry imposes the following relations between their OPE coefficients,⁵

$$\lambda_{(Q\bar{Q}\mathcal{O})_{\ell+1}}^2 = \frac{(\Delta + \ell)}{4(\Delta + \ell + 1)} \lambda_{\mathcal{O}}^2, \quad (4.26)$$

$$\lambda_{(Q\bar{Q}\mathcal{O})_{\ell-1}}^2 = \frac{(\Delta - \ell - 2)}{4(\Delta - \ell - 1)} \lambda_{\mathcal{O}}^2, \quad (4.27)$$

$$\lambda_{Q^2\bar{Q}^2\mathcal{O}}^2 = \frac{(\Delta + \ell)(\Delta - \ell - 2)}{16(\Delta + \ell + 1)(\Delta - \ell - 1)} \lambda_{\mathcal{O}}^2. \quad (4.28)$$

Note that $\lambda_{(Q\bar{Q}\mathcal{O})_{\ell-1}}^2$ and $\lambda_{Q^2\bar{Q}^2\mathcal{O}}^2$ vanish when $\Delta = \ell + 2$, consistent with shortening of the superconformal multiplet.

Meanwhile, the $\Phi \times \Phi$ OPE can only contain operators which are killed by \bar{Q} . First and foremost, we have the chiral primary Φ^2 , whose dimension is exactly $2d$, by virtue of the relation between dimension and R -charge for chiral operators. All other operators are \bar{Q} -descendants. Schematically,

$$\Phi \times \Phi \sim \Phi^2 + \sum_{\ell=2,4,\dots} \bar{Q}\mathcal{O}_\ell + \sum_{\mathcal{O}} \bar{Q}^2\mathcal{O}. \quad (4.29)$$

The operators \mathcal{O}_ℓ transform in $(\frac{\ell}{2}, \frac{\ell-1}{2})$ representations of the Lorentz group $\text{SO}(4) \cong \text{SU}(2) \times \text{SU}(2)$, and satisfy the BPS shortening condition $\bar{Q}_{\dot{\alpha}} \mathcal{O}_\ell^{\dot{\alpha}\dot{\alpha}_3 \dots \dot{\alpha}_\ell, \alpha_1 \dots \alpha_\ell} = 0$. The product $\bar{Q}\mathcal{O}_\ell$ is then a spin-

⁵The difference in normalization from the formulae in [57] is due to our different convention for conformal blocks Eq. (4.6).

ℓ operator, which is required by the superconformal algebra to have dimension $2d + \ell$. Finally, the remaining operators $\overline{Q}^2 \mathcal{O}$ are not protected by a BPS condition, and can have any dimension satisfying $\Delta \geq |2d - 3| + 3 + \ell$. Note that when $d < 3/2$, a gap in dimensions exists between the protected operators $\Phi^2, \overline{Q} \mathcal{O}_\ell$ and the non-protected operators $\overline{Q}^2 \mathcal{O}$. In contrast to the situation for $\Phi \times \Phi^\dagger$, each conformal primary in Eq. (4.29) appears with an independent coefficient — there are no additional relations imposed by supersymmetry among operators in $\Phi \times \Phi$.

Because of the $U(1)_R$ symmetry, crossing symmetry of the four-point function $\langle \Phi \Phi^\dagger \Phi \Phi^\dagger \rangle$ is a special case of crossing symmetry for $SO(2)$. Note that the antisymmetric tensor representation of $SO(2)$ is the trivial representation, so that we may equivalently write S^- (odd-spin singlets) for A^- (odd-spin antisymmetric tensors). We are also free to multiply the sum rule by any invertible matrix without changing its content.⁶ Consequently, we can rewrite Eq. (4.14) for $SO(2)$ as

$$\sum_{S^\pm} \lambda_{\mathcal{O}}^2 \begin{pmatrix} F_{\Delta,\ell} \\ (-)^\ell F_{\Delta,\ell} \\ (-)^\ell H_{\Delta,\ell} \end{pmatrix} + \sum_{T^+} 2\lambda_{\mathcal{O}}^2 \begin{pmatrix} 0 \\ F_{\Delta,\ell} \\ -H_{\Delta,\ell} \end{pmatrix} = 0. \quad (4.30)$$

In our superconformal four-point function $\langle \Phi \Phi^\dagger \Phi \Phi^\dagger \rangle$, the S^\pm terms will come from the OPE (4.25), while the T^+ terms come from (4.29). Making use of the relations between OPE coefficients Eq. (4.26), the specialization of Eq. (4.30) to the superconformal case is

$$\sum_{S^\pm} \lambda_{\mathcal{O}}^2 \begin{pmatrix} \mathcal{F}_{\Delta,\ell} \\ \tilde{\mathcal{F}}_{\Delta,\ell} \\ \tilde{\mathcal{H}}_{\Delta,\ell} \end{pmatrix} + \sum_{T_{\text{BPS}}^+} \lambda_{\mathcal{O}}^2 \begin{pmatrix} 0 \\ F_{2d+\ell,\ell} \\ -H_{2d+\ell,\ell} \end{pmatrix} + \sum_{T_{\text{non-BPS}}^+} \lambda_{\mathcal{O}}^2 \begin{pmatrix} 0 \\ F_{\Delta,\ell} \\ -H_{\Delta,\ell} \end{pmatrix} = 0, \quad (4.31)$$

where

$$\begin{aligned} \mathcal{F}_{\Delta,\ell} \equiv & F_{\Delta,\ell} + \frac{(\Delta + \ell)}{4(\Delta + \ell + 1)} F_{\Delta+1,\ell+1} + \frac{(\Delta - \ell - 2)}{4(\Delta - \ell - 1)} F_{\Delta+1,\ell-1} \\ & + \frac{(\Delta + \ell)(\Delta - \ell - 2)}{16(\Delta + \ell + 1)(\Delta - \ell - 1)} F_{\Delta+2,\ell}. \end{aligned} \quad (4.32)$$

⁶Specifically, we will replace the middle row with itself plus twice the top row, and the top row with the middle row.

In addition, $\tilde{\mathcal{F}}$ is \mathcal{F} with odd spins flipped $F_{\Delta,\ell} \rightarrow (-)^\ell F_{\Delta,\ell}$ throughout, and $\tilde{\mathcal{H}}$ is $\tilde{\mathcal{F}}$ with $F_{\Delta,\ell} \rightarrow H_{\Delta,\ell}$. The set T_{BPS}^+ consists of the BPS operators appearing in $\Phi \times \Phi$, namely Φ^2 and $\overline{Q}\mathcal{O}_\ell$ for $\ell \in \{2, 4, \dots\}$. $T_{\text{non-BPS}}^+$ consists of the remaining operators in $\Phi \times \Phi$. In going from Eq. (4.30) to Eq. (4.31), we have removed the factors of 2 in front of the symmetric tensor contributions because their conventional normalization differs between $\text{SO}(2)$ and $U(1)$, $[\lambda_{T^+}^2]_{U(1)} = 2[\lambda_{T^+}^2]_{\text{SO}(2)}$.

Superconformal $\text{SU}(N)$ Crossing Relations

It is straightforward to generalize this analysis to the case of a scalar superconformal primary Φ_i transforming as a fundamental under an $\text{SU}(N)$ global symmetry. The index structure of the OPE is the same as is given in Eqs. (4.21) and (4.22), but with the additional constraints imposed by supersymmetry discussed above. Note that now both BPS and non-BPS odd-spin operators can appear in the $\Phi_i \times \Phi_j$ OPE as $\text{SU}(N)$ antisymmetric tensors. Including these constraints, the six-fold sum rule of Eq. (4.23) becomes

$$\begin{aligned} \sum_{S^\pm} \lambda_{\mathcal{O}}^2 \mathcal{V}_{\Delta,\ell}^{S^\pm} + \sum_{\text{Ad}^\pm} \lambda_{\mathcal{O}}^2 \mathcal{V}_{\Delta,\ell}^{\text{Ad}^\pm} + \sum_{T_{\text{BPS}}^+} \lambda_{\mathcal{O}}^2 V_{2d+\ell,\ell}^{T^+} + \sum_{A_{\text{BPS}}^-} \lambda_{\mathcal{O}}^2 V_{2d+\ell,\ell}^{A^-} \\ + \sum_{T_{\text{non-BPS}}^+} \lambda_{\mathcal{O}}^2 V_{\Delta,\ell}^{T^+} + \sum_{A_{\text{non-BPS}}^-} \lambda_{\mathcal{O}}^2 V_{\Delta,\ell}^{A^-} = 0, \end{aligned} \quad (4.33)$$

where

$$\mathcal{V}^{S^\pm} = \begin{pmatrix} \mathcal{F} \\ \mathcal{H} \\ \tilde{\mathcal{F}} \\ \tilde{\mathcal{H}} \\ 0 \\ 0 \end{pmatrix}, \quad \mathcal{V}^{\text{Ad}^\pm} = \begin{pmatrix} (1 - \frac{1}{N})\mathcal{F} \\ -(1 + \frac{1}{N})\mathcal{H} \\ -\frac{1}{N}\tilde{\mathcal{F}} \\ -\frac{1}{N}\tilde{\mathcal{H}} \\ \tilde{\mathcal{F}} \\ \tilde{\mathcal{H}} \end{pmatrix}, \quad V^{T^+} = \begin{pmatrix} 0 \\ 0 \\ F \\ -H \\ F \\ -H \end{pmatrix}, \quad V^{A^-} = \begin{pmatrix} 0 \\ 0 \\ F \\ -H \\ -F \\ H \end{pmatrix}. \quad (4.34)$$

4.2.4 Bounds from Crossing Relations

Crossing symmetry of four-point functions encodes an infinite number of relations between OPE coefficients — one for each value of the conformal cross-ratios u and v . In [54] a general method was outlined for extracting bounds on CFT data using these relations, together with the constraints of unitarity. We will now review this method for the simplest case of a real scalar ϕ of dimension d . Subsequently, we will discuss how the original method can be improved using semidefinite programming.

Suppose we would like to bound the OPE coefficient of a particular operator \mathcal{O}_0 of dimension Δ_0 and spin ℓ_0 appearing in $\phi \times \phi$. The first step is to isolate $\lambda_{\mathcal{O}_0}^2$ on one side of the sum rule Eq. (4.9),

$$\lambda_{\mathcal{O}_0}^2 F_{\Delta_0, \ell_0}(u, v) = -F_{0,0}(u, v) - \sum_{\mathcal{O} \neq \mathcal{O}_0} \lambda_{\mathcal{O}}^2 F_{\Delta, \ell}(u, v). \quad (4.35)$$

We can obtain different expressions for $\lambda_{\mathcal{O}_0}^2$ in terms of the other OPE coefficients by evaluating Eq. (4.35) at different values of u and v . We could also take some number of u - and v -derivatives first, and then evaluate. And in general, we can apply any linear functional α to both sides,

$$\lambda_{\mathcal{O}_0}^2 \alpha(F_{\Delta_0, \ell_0}) = -\alpha(F_{0,0}) - \sum_{\mathcal{O} \neq \mathcal{O}_0} \lambda_{\mathcal{O}}^2 \alpha(F_{\Delta, \ell}). \quad (4.36)$$

A key insight of [54] is that the functions $F_{\Delta, \ell}$ share certain positivity properties, so that it's sometimes possible to find a linear functional α such that

$$\alpha(F_{\Delta_0, \ell_0}) = 1, \quad \text{and} \quad (4.37)$$

$$\alpha(F_{\Delta, \ell}) \geq 0, \quad \text{for all other (non-unit) operators in the spectrum.} \quad (4.38)$$

Eq. (4.37) is simply a normalization condition, but to satisfy Eq. (4.38) one must choose α carefully.

If α satisfies these constraints, then since the $\lambda_{\mathcal{O}}^2$ are positive by unitarity, Eq. (4.36) becomes an

upper bound on $\lambda_{\mathcal{O}_0}^2$,

$$\lambda_{\mathcal{O}_0}^2 = -\alpha(F_{0,0}) - \sum_{\mathcal{O} \neq \mathcal{O}_0} \text{pos.} \times \text{pos.} \leq -\alpha(F_{0,0}). \quad (4.39)$$

The space of viable α 's depends on precisely what assumptions one makes about the spectrum of the CFT. If one makes an assumption about the spectrum of operator dimensions that makes it easier to satisfy Eq. (4.38) (e.g., all scalars have a dimension greater than some Δ_{\min}) and then finds a linear functional α such that the bound of Eq. (4.39) violates the unitarity constraint $\lambda_{\mathcal{O}_0}^2 \geq 0$, one can rule out that assumption about the spectrum.

Now, to make the bound (4.39) as strong as possible, we should minimize $-\alpha(F_{0,0})$ over the set \mathcal{S} of all α satisfying the constraints (4.37, 4.38). These constraints carve out a convex subset of the space of linear functionals, so the task of determining the best α is an infinite-dimensional *convex optimization problem*. It would be extremely interesting to develop analytical techniques for finding solutions. However, the most successful approaches to date, including the one we present here, involve simplifying the problem to make it tractable on a computer, and then determining solutions numerically.

Putting our optimization problem on a computer requires surmounting two difficulties:

1. The search space \mathcal{S} of α 's satisfying Eqs. (4.37, 4.38) is infinite dimensional.
2. The number of constraints $\alpha(F_{\Delta,\ell}) \geq 0$ is infinite — there's one for each Δ, ℓ .

The first difficulty is easy enough to address: we can restrict to a finite-dimensional subspace \mathcal{W} of linear functionals. Then, minimizing $-\alpha(F_{0,0})$ over all $\alpha \in \mathcal{W} \cap \mathcal{S}$ will give a possibly sub-optimal, but still valid bound $\lambda_{\mathcal{O}_0}^2 \leq -\alpha(F_{0,0})$. The choice of \mathcal{W} is somewhat arbitrary, and it would be interesting to explore a wider variety of functionals than we do here. Following [54–60], we will simply take linear combinations of derivatives around the symmetric point $z = \bar{z} = 1/2$.

That is, we define \mathcal{W}_k to be the space of functionals

$$\alpha : F(z, \bar{z}) \mapsto \sum_{m+n \leq 2k} a_{mn} \partial_z^m \partial_{\bar{z}}^n F(1/2, 1/2), \quad (4.40)$$

with real coefficients a_{mn} . This choice is computationally convenient, and will prove useful in our solution to the second difficulty in a moment. One hopes that as we increase k to include more and more derivatives, our search will cover more and more of \mathcal{S} , and our bound will converge to the optimal one.

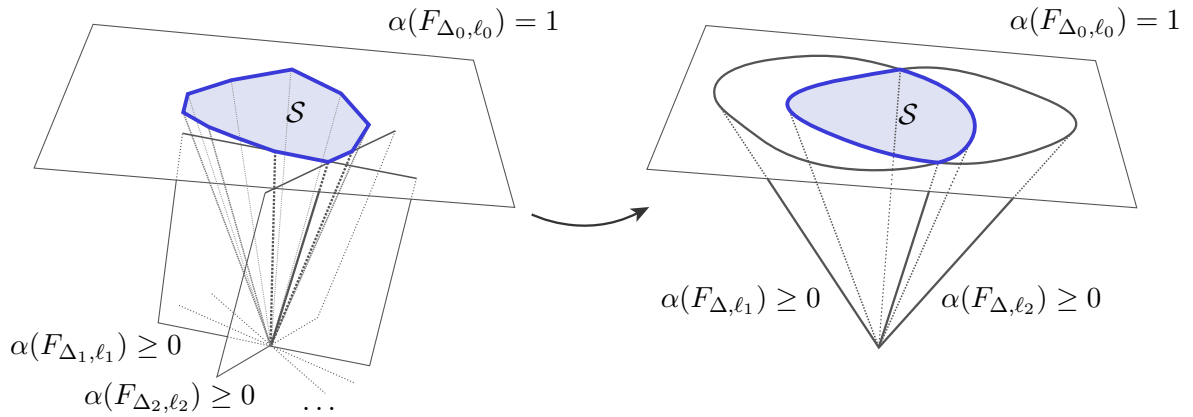


Figure 4.1: The ‘search space’ \mathcal{S} (shown in blue) is the intersection of the hyperplane $\alpha(F_{\Delta_0, \ell_0}) = 1$ with the convex cone of linear functionals α satisfying $\alpha(F_{\Delta, \ell}) \geq 0$ for all $\{\Delta, \ell\}$ in the spectrum. Previous methods discretized Δ to some finite set $\{\Delta_i\}$, thus approximating \mathcal{S} as an intersection of a finite number of hyperplanes and half-spaces (left). Our approach is to approximate \mathcal{S} as the intersection of a smaller number of curved spaces — specifically cones of semidefinite matrices (right). Such intersections are sometimes called *spectrahedra*.

The second difficulty is more problematic. Since angular momentum ℓ is discrete, it’s reasonable to include constraints with $\ell = 0, 1, \dots, L$, for some large L . But the dimension Δ can vary continuously, and the constraints $\alpha(F_{\Delta, \ell}) \geq 0$ carve out a complicated shape \mathcal{S} inside \mathcal{W} as Δ varies. The computer has to know about this shape, which means we must encode it with some finite amount of data. The approach used in [54–60] is to approximate the shape by a convex polytope — namely discretize Δ to lie in some finite set $\{\Delta_i\}$, so that the constraints $\alpha(F_{\Delta_i, \ell}) \geq 0$ become a finite number of linear inequalities for α . Then the problem of minimizing $-\alpha(F_{0,0})$

becomes a *linear programming* problem, which can be solved by jumping from vertex to vertex on the boundary of the polytope, following the direction of steepest descent. As one makes the set $\{\Delta_i\}$ larger, the approximation of \mathcal{S} as a polytope gets more and more refined, and the solution should converge to the correct one.

This method can be quite powerful if one chooses the $\{\Delta_i\}$ carefully. However, some basic tensions limit how far it can be pushed. For example, consider increasing k to obtain a stronger bound. At higher k , the space \mathcal{W}_k can include wilder linear functionals, and one must include more Δ_i to ensure that a constraint $\alpha(F_{\Delta,l}) \geq 0$ isn't violated. However, the running time of the usual search algorithm is *cubic* in the number of constraints, which means that computations become quickly unwieldy.

Our approach in the present paper is to approximate \mathcal{S} with a different kind of shape that is more efficient to encode than a polytope, one that naturally respects the properties of conformal blocks (specifically the differential equation that they satisfy), and also admits fast searches. In the process, we will do away with the discretization $\Delta \in \{\Delta_i\}$ entirely.

4.2.5 Semidefinite Programming

Semidefinite programs (SDPs) [106] are linear optimization problems that can contain positive-semidefiniteness constraints for matrices, along with the usual linear inequalities included in linear programs. As we'll see momentarily, positive-semidefiniteness lets us express the condition that a collection of polynomials be nonnegative for all values of their arguments. This is useful for us because there is a systematic approximation for the derivatives of $F_{\Delta,l}$ in terms of polynomials. Specifically, there exist positive functions $\chi_\ell(\Delta)$ and polynomials $P_\ell^{m,n}(\Delta)$ such that

$$\partial_z^m \partial_{\bar{z}}^n F_{\Delta,\ell}(1/2, 1/2) \approx \chi_\ell(\Delta) P_\ell^{mn}(\Delta), \quad (4.41)$$

where the approximation can be made arbitrarily good, at the cost of increasing the degree of P_ℓ^{mn} . The details of this approximation, which follows from the differential equation for conformal blocks

along with some basic facts about hypergeometric functions, are explained in appendix 4.A.

For now, let us assume that such an approximation exists, and understand how to phrase our problem as an SDP. We will write $F_\ell^{mn}(\Delta) \equiv \partial_z^m \partial_{\bar{z}}^n F_{\Delta,\ell}(1/2, 1/2)$ for brevity. Once again, we would like to minimize $-a_{mn} F_0^{mn}(0)$ subject to the constraints

$$a_{mn} F_{\ell_0}^{mn}(\Delta_0) = 1, \quad (4.42)$$

$$a_{mn} F_\ell^{mn}(\Delta) \geq 0 \quad \text{for } \Delta \geq \Delta_\ell, \text{ for all } 0 \leq \ell \leq L, \quad (4.43)$$

where Δ_ℓ is a lower bound on Δ depending on the spin ℓ .

Using Eq. (4.41) along with the fact that $\chi_\ell(\Delta)$ is positive, Eq. (4.43) becomes the statement that each polynomial $a_{mn} P_\ell^{mn}(\Delta_\ell(1+x))$ is nonnegative on the interval $x \in [0, \infty)$. Such statements are naturally written in terms of positive-semidefinite matrices, a fact which is well-known in the optimization literature and has been exploited to solve a wide variety of problems (see, e.g., [107]). The rewriting proceeds as follows. Firstly, a theorem due to Hilbert [108] states that a polynomial $p(x)$ is nonnegative on $[0, \infty)$ if and only if

$$p(x) = f(x) + xg(x), \quad (4.44)$$

where both $f(x)$ and $g(x)$ are sums of squares of polynomials. Now suppose f and g have degrees $2d$ and $2d'$ respectively, and let $[x]_d$ denote the vector with entries $(1, x, \dots, x^d)$. If $f(x)$ is a sum of squares of polynomials with coefficients $\mathbf{c}_i = (c_{i0}, \dots, c_{id})$, then we have

$$f(x) = \sum_i (\mathbf{c}_i^T [x]_d)^2 = [x]_d^T \left(\sum_i \mathbf{c}_i \mathbf{c}_i^T \right) [x]_d = [x]_d^T A [x]_d, \quad (4.45)$$

where $A \equiv \sum_i \mathbf{c}_i \mathbf{c}_i^T$ is positive-semidefinite. Conversely, any positive-semidefinite matrix A admits a Cholesky decomposition $A = \sum_i \mathbf{c}_i \mathbf{c}_i^T$, so that $[x]_d^T A [x]_d$ is a sum of squares. Thus, the condition that $p(x)$ be nonnegative on $[0, \infty)$ can be written

$$p(x) = [x]_d^T A [x]_d + x([x]_{d'}^T B [x]_{d'}), \quad \text{with} \quad A, B \succeq 0, \quad (4.46)$$

where the \succeq symbol means ‘positive-semidefinite.’

Returning to OPE bounds, we now have the following presentation of our convex optimization problem as an SDP: minimize $-a_{mn}F_0^{mn}(0)$, subject to the constraints

$$a_{mn}F_{\ell_0}^{mn}(\Delta_0) = 1, \quad (4.47)$$

$$a_{mn}F_{\ell}^{mn}(\Delta_{\ell}(1+x)) = [x]_{d_{\ell}}^T A_{\ell}[x]_{d_{\ell}} + x([x]_{d'_{\ell}}^T B_{\ell}[x]_{d'_{\ell}}) \quad \text{for } 0 \leq \ell \leq L, \quad (4.48)$$

$$A_{\ell}, B_{\ell} \succeq 0 \quad \text{for } 0 \leq \ell \leq L. \quad (4.49)$$

There are numerous advantages to this formulation. Firstly, we avoid discretizing the set of operator dimensions Δ , and thus evade the trade-off between refining $\{\Delta_i\}$ and improving the running time. Further, small and large Δ are accounted for equally well, so there is no need for separate checks on the asymptotic behavior of $\alpha(F_{\Delta,\ell})$ at large dimensions. Most importantly, there exist efficient algorithms for solving semidefinite programs using interior point methods, with some excellent implementations (see appendix 4.B). Their complexity scales much less sharply with the dimension of the search space than the linear programming algorithms used in [54–60]. Consequently, we have been able to push the previous state-of-the-art searches from 55 dimensions to almost 400 dimensions in some cases.

4.2.6 Generalizations for Global Symmetries

While we specialized the above discussion to the case of the singlet sum rule Eq. (4.9), it is straightforward to modify it for situations with global symmetries. E.g., if we wish to place a bound on the OPE coefficient of an S^+ operator appearing in the $\text{SO}(N)$ sum rule of Eq. (4.14),

we should look for a vectorial linear functional α satisfying

$$\alpha \begin{pmatrix} 0 \\ F_{\Delta_0, \ell_0} \\ H_{\Delta_0, \ell_0} \end{pmatrix} = 1, \quad (4.50)$$

$$\alpha \begin{pmatrix} 0 \\ F_{\Delta, \ell} \\ H_{\Delta, \ell} \end{pmatrix} \geq 0, \quad \text{for all other (non-unit) operators in } S^+, \quad (4.51)$$

$$\alpha \begin{pmatrix} F_{\Delta, \ell} \\ (1 - \frac{2}{N}) F_{\Delta, \ell} \\ -(1 + \frac{2}{N}) H_{\Delta, \ell} \end{pmatrix} \geq 0, \quad \text{for all operators in } T^+, \text{ and} \quad (4.52)$$

$$\alpha \begin{pmatrix} -F_{\Delta, \ell} \\ F_{\Delta, \ell} \\ -H_{\Delta, \ell} \end{pmatrix} \geq 0, \quad \text{for all operators in } A^-. \quad (4.53)$$

Any such linear functional then leads to the upper bound

$$\lambda_{\mathcal{O}_0}^2 \leq -\alpha \begin{pmatrix} 0 \\ F_{0,0} \\ H_{0,0} \end{pmatrix}. \quad (4.54)$$

The modification for alternatively placing bounds on the OPE coefficients of T^+ or A^- operators should be clear. As in the singlet case, we can also rule out an assumption about the spectrum of operator dimensions by making the assumption and then finding a linear functional that leads to a violation of the unitarity constraint $\lambda_{\mathcal{O}_0}^2 \geq 0$.

Similarly, we can bound the OPE coefficient of an S^\pm operator appearing in the $SU(N)$

sum rule of Eq. (4.23) by finding an α satisfying

$$\alpha \left(V_{\Delta_0, \ell_0}^{S^\pm} \right) = 1, \quad (4.55)$$

$$\alpha \left(V_{\Delta, \ell}^I \right) \geq 0, \quad \text{for all other (non-unit) operators in the spectrum,} \quad (4.56)$$

where $I = \{S^\pm, \text{Ad}^\pm, T^+, A^-\}$. This leads to the upper bound

$$\lambda_{\mathcal{O}_0}^2 \leq -\alpha \left(V_{0,0}^{S^+} \right). \quad (4.57)$$

The appropriate generalization of this logic for placing bounds on operators in other $\text{SU}(N)$ representations, and also for obtaining bounds using the superconformal sum rules given in Eqs. (4.31) and (4.33), should be clear.

In all of these situations, the task of numerically finding the optimal α can be recast in terms of a semidefinite program. Similar to what we described in the previous section, to do this we use the fact that derivatives of any of the functions $\{F_{\Delta, \ell}, H_{\Delta, \ell}, \mathcal{F}_{\Delta, \ell}, \mathcal{H}_{\Delta, \ell}, \tilde{\mathcal{F}}_{\Delta, \ell}, \tilde{\mathcal{H}}_{\Delta, \ell}\}$ at $(1/2, 1/2)$ can be arbitrarily-well approximated by positive functions times polynomials in Δ . The details of these approximations can be found in appendix 4.A.

4.2.7 Coincidence Between $\text{SU}(N)$ and $\text{SO}(2N)$ Singlet Bounds

In the course of running the above algorithm, we found that our bounds on singlet operators appearing in an OPE between $\text{SU}(N)$ fundamentals were numerically identical to bounds on singlets appearing in an OPE between $\text{SO}(2N)$ fundamentals. This exact coincidence is surprising given the rather different structure of the crossing symmetry constraints. It hadn't been previously observed because $\text{SU}(N)$ computations were too difficult to perform with previous techniques. In this section, we'll discuss the relations between those bounds in more detail.

Let us consider more generally a CFT with global symmetry group \mathcal{G} . Suppose we want to obtain a dimension bound on a singlet scalar operator entering a given OPE. The \mathcal{G} crossing symmetry constraints produce a bound $\Delta_{\mathcal{G}}(d)$. Now consider a subgroup $\mathcal{H} \subset \mathcal{G}$ and repeat the

procedure. This time, the \mathcal{H} crossing symmetry constraints will produce a bound $\Delta_{\mathcal{H}}(d)$. At this point we must distinguish two cases: 1) all \mathcal{H} -singlets are also \mathcal{G} -singlets, 2) some nontrivial representation of \mathcal{G} , once decomposed with respect to the subgroup, contains \mathcal{H} -singlets. In the first case we can immediately conclude

$$\Delta_{\mathcal{G}}(d) \leq \Delta_{\mathcal{H}}(d) \quad (\mathcal{G}\text{-bound stronger}). \quad (4.58)$$

The above inequality is clear: there are no CFT's with global symmetry \mathcal{H} where the first scalar singlet operator entering a given OPE has dimension larger than $\Delta_{\mathcal{H}}(d)$. Thus in particular there are no CFT's with a larger global symmetry.

An example of such a group and subgroup is given precisely by $SU(N) \subset SO(2N)$. In the decomposition with respect to the subgroup, the only singlets come from $SO(2N)$ -singlets: the symmetric tensor goes to a symmetric tensor and an adjoint while the antisymmetric tensor goes to an antisymmetric tensor and an adjoint. Thus, it is natural to expect the triple sum rule Eq. (4.14) to give a bound stronger than or equal to the sextuple sum rule Eq. (4.23). Indeed, one can verify this explicitly at the level of the optimization problem for α .

To prove the equality of $SU(N)$ and $SO(2N)$ bounds one should also show that whenever a linear functional satisfying Eqs. (4.50-4.53) exists, it is possible to construct a second linear functional satisfying Eq. (4.55). Unfortunately, we have not been able to find an analytic proof of this result. However, we find numerically that it is always possible – it would be good in future studies to gain a deeper understanding of why this is the case.

In the case 2) the two bounds are unrelated, since the \mathcal{H} -bound could in principle be determined by representations coming from the decomposition of nontrivial representations of the larger symmetry group. This is the case for $SO(N)$ and $SO(N')$ or $SU(N)$ and $SU(N')$, with $N > N'$. In these examples we numerically observe behavior opposite to (4.58).

4.3 Bounds on Operator Dimensions

4.3.1 General Theories

As a first application of our semidefinite programming algorithm, let us reproduce the singlet dimension bound first derived in [54], and later improved in [55]. We let ϕ be a real scalar of dimension d in a general CFT, and seek to place an upper bound on the dimension of ϕ^2 , the lowest dimension scalar appearing in $\phi \times \phi$. The procedure is precisely as described in section 4.2.4. In figure 4.2, we show the resulting bounds for $k = 2, \dots, 11$, with $k = 10$ (a 55-dimensional search-space) being the previous state-of-the-art. We find perfect agreement with older linear programming-based calculations for each $k = 2, \dots, 10$.

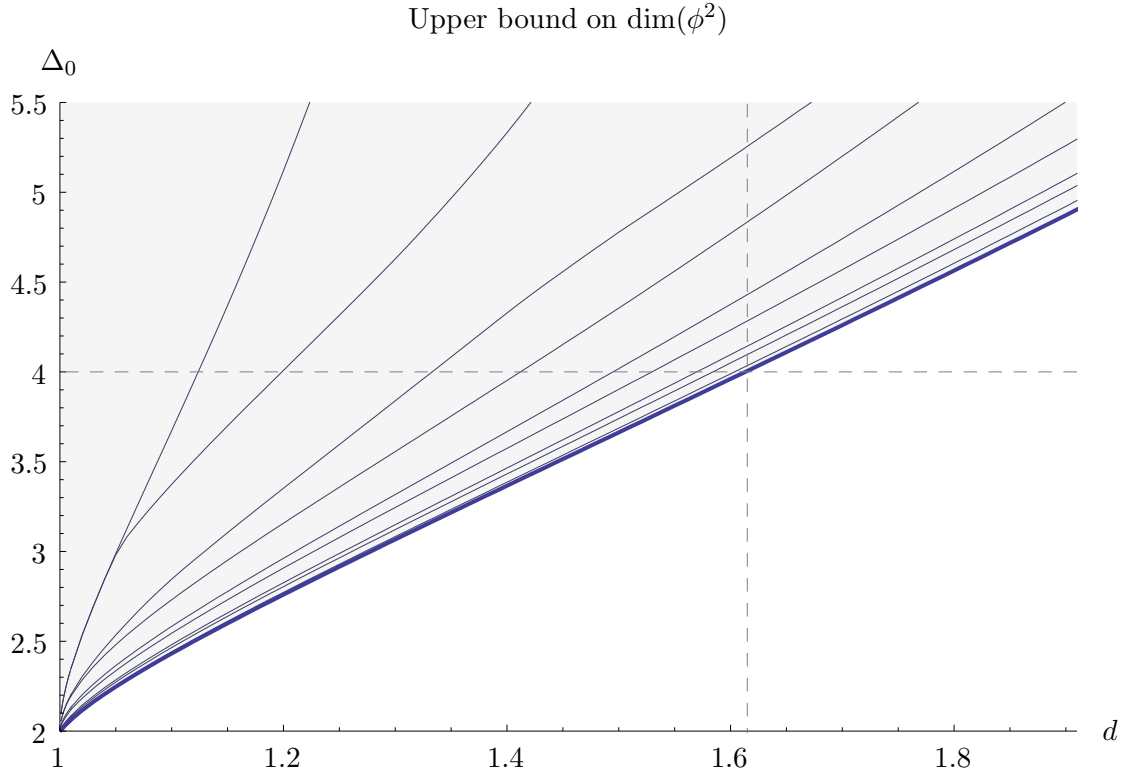


Figure 4.2: An upper bound on the dimension of ϕ^2 , the lowest dimension scalar appearing in $\phi \times \phi$. Curves for $k = 2, \dots, 11$ are shown, with the $k = 11$ bound being the strongest.

The curves appear to converge at large k , which is perhaps indicative that they are approaching the best possible bound given our assumptions (referred to as $f_\infty(d)$ in [55]).⁷ We will see this kind of convergence in many other plots in this paper. An approximate fit to the strongest ($k = 11$) bound is given by⁸

$$\dim(\phi^2) \leq 2 + 3.006\epsilon + 0.160(1 - e^{-20\epsilon}), \quad (4.59)$$

where $d = 1 + \epsilon$, with ϵ between 0 and 1. Notice that the behavior for both small and large ϵ is approximately linear. The bound crosses $\dim(\phi^2) = 4$ around $d \approx 1.61$.

4.3.2 Singlet Operators in $\text{SO}(N)$ and $\text{SU}(N)$ Theories

We can also place bounds on the lowest dimension singlet appearing in $\phi_i \times \phi_j$, where ϕ_i transforms as a vector of an $\text{SO}(N)$ global symmetry. The procedure is as described in section 4.2.6, where we must assume that $\Delta > \Delta_{\min}$ for all scalars in S^+ , and then scan over Δ_{\min} to obtain a dimension bound. Recall from section 4.2.7 that our bounds on singlets of $\text{SU}(N)$ turn out to be identical to those for singlets of $\text{SO}(2N)$. Hence, we will present all SU and SO singlet bounds together, with even values of N standing for both $\text{SO}(N)$ and $\text{SU}(N/2)$.

Previous attempts to compute bounds for theories with global symmetries have been somewhat hindered by the need to optimize over very high-dimensional spaces. Since the vectorial sum rule Eq. (4.14) has three components, a given k corresponds to

$$\frac{k(k+1)}{2} \times 3 \quad (4.60)$$

different linear functionals. The linear programming methods implemented so far are essentially limited to a search space dimension that is not much larger than ~ 50 , or $k \sim 5$ for $\text{SO}(N)$. Worse,

⁷However, since the full optimization problem involves an infinite-dimensional search space, it's always possible a new search direction could open up at higher k . Fully establishing convergence would require more detailed analysis than we do here.

⁸While it gives a good description of the shape, we have chosen this functional form somewhat arbitrarily; it is possible that a different basis of functions should be used when describing the optimal bound.

$SU(N)$ vectorial sum rules have six components, making them even harder to explore. However, our semidefinite programming algorithm appears to have few problems with large search spaces, and we will present most of our bounds up to $k = 11$, regardless of the type of global symmetry group.

As an example, figure 4.3 shows a bound on the lowest dimension singlet in theories with an $SU(2)$ or $SO(4)$ global symmetry.⁹ This bound is particularly interesting for conformal technicolor models, as we will discuss in detail in the following section. Notice again that the curves start to converge at large k . An approximate fit to the strongest ($k = 11$) bound is given by

$$\dim(|\phi|^2) \leq 2 + 3.119\epsilon + 0.398(1 - e^{-12\epsilon}), \quad (4.61)$$

where $d = 1 + \epsilon$, with ϵ between 0 and 1. This bound crosses $\Delta_0 = 4$ around $d \approx 1.52$.

Figure 4.4 shows dimension bounds for $SO(N)$ with $N = 2, \dots, 14$ and $SU(N)$ with $N = 2, \dots, 7$. The strongest bound corresponds to the global symmetry group $SO(2) \cong U(1)$, and the bounds weaken as N increases. One might naïvely expect a larger symmetry group to produce a stronger bound. For instance, a theory with an $SO(N)$ symmetry certainly also has an $SO(N-1)$ symmetry, so why shouldn't all bounds from the former apply to the latter? However, as discussed in section 4.2.7, the problem we are solving actually changes with N , and this turns out to be a more important effect than the enhanced symmetry. Note that the lowest dimension singlet under an $SO(N-1)$ subgroup of $SO(N)$ is not necessarily a singlet at all under the full $SO(N)$. Thus, $SO(N)$ bounds for larger N apply to the operator with lowest dimension among a more restricted class of operators, and consequently can be weaker.

⁹Note that to compute the $SO(4)$ bound, we have only used the triple sum rule of Eq. (4.14). It is straightforward to verify that including the fourth sum rule of Eq. (4.18) leads to a redundant set of constraints, and is therefore unnecessary.

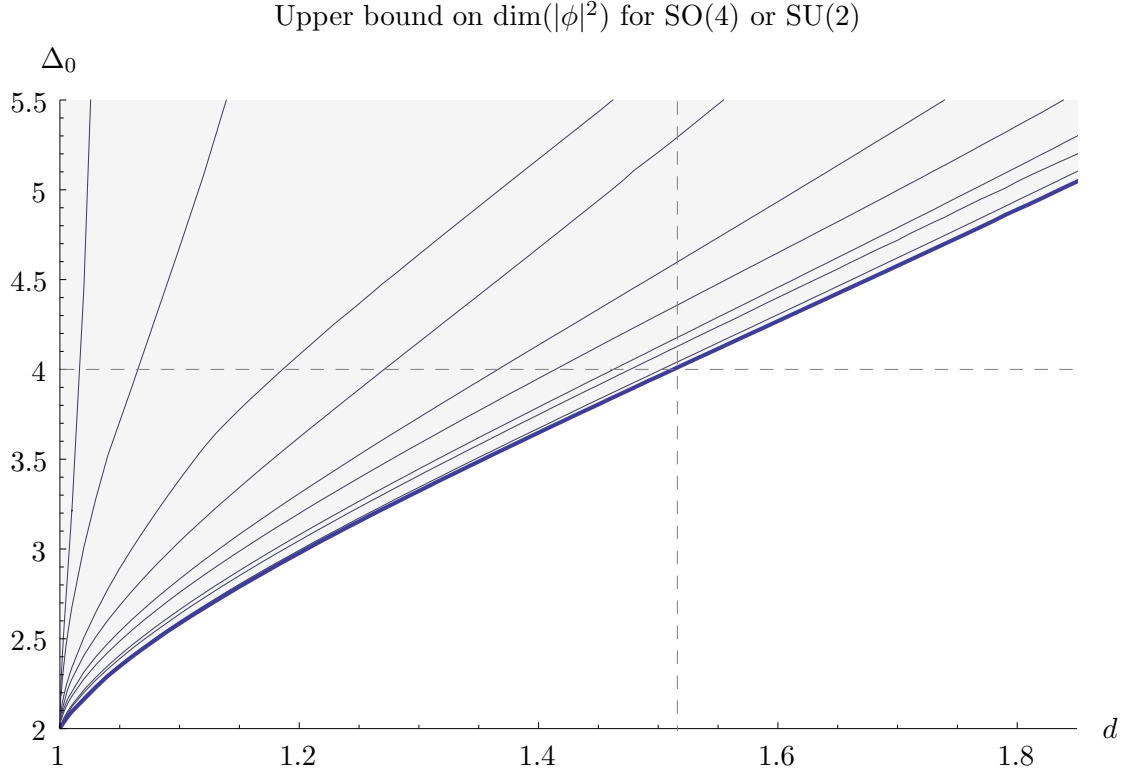


Figure 4.3: An upper bound on the dimension of $\phi^\dagger\phi$, the lowest dimension singlet scalar appearing in $\phi^\dagger \times \phi$, where ϕ transforms in the fundamental representation of an SO(4) or an SU(2) global symmetry. Curves are shown for $k = 2, \dots, 11$. The bounds for SO(4) and SU(2) are identical in each case. The strongest bound crosses $\Delta_0 = 4$ around $d = 1.52$.

Implications for Conformal Technicolor

Let us briefly discuss some phenomenological implications of the bounds presented in figures 4.3 and 4.4. A more detailed discussion of these implications will also appear in [109], and our analysis draws heavily on the previous discussions of [8–11, 54, 59, 60], as well as the recent talk of [110].

Arguably the most interesting operator dimension in the Standard Model is $\dim(H^\dagger H)$, the dimension of the Higgs mass operator, where H transforms as a bifundamental under $SU(2)_L \times U(1)_Y \subset SU(2)_L \times SU(2)_R$. In a weakly-coupled theory with a scalar Higgs, this dimension is approximately 2, which leads to the hierarchy problem and its associated puzzles.

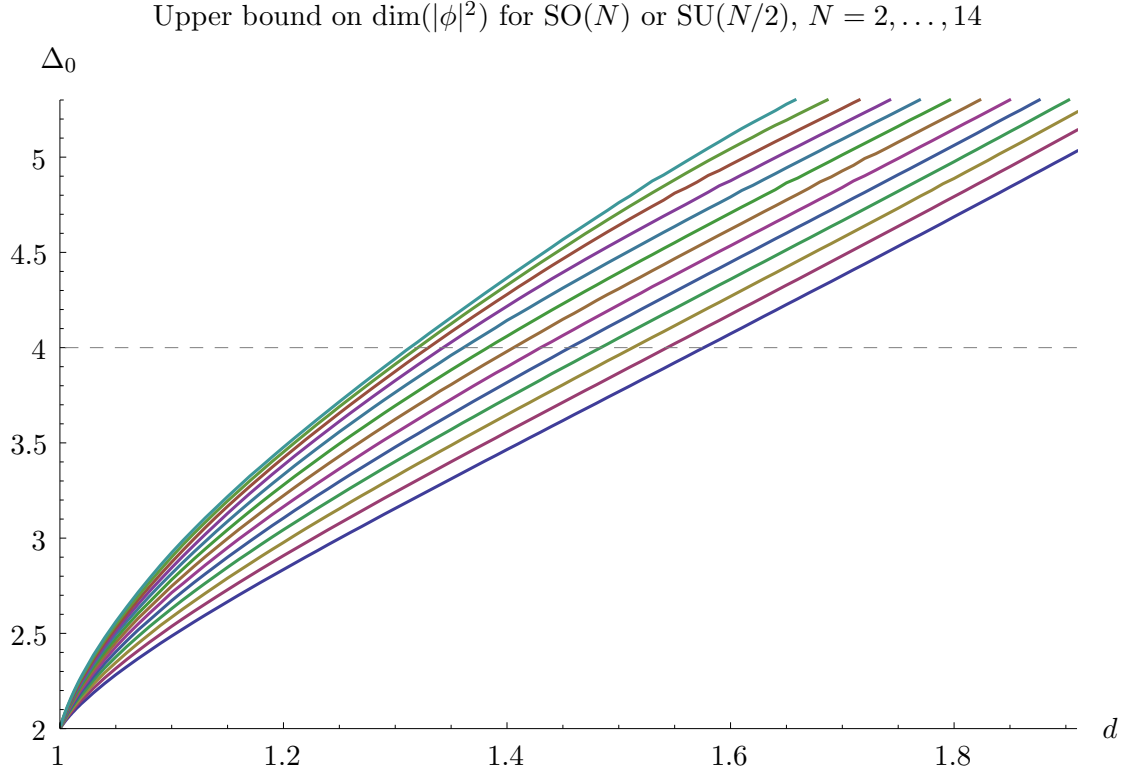


Figure 4.4: An upper bound on the dimension of $|\phi|^2$, the lowest dimension singlet scalar appearing in $\phi^T \times \phi$ (or $\phi^\dagger \times \phi$), where ϕ transforms in the fundamental representation of an $\text{SO}(N)$ global symmetry or an $\text{SU}(N/2)$ global symmetry (when $N \geq 4$ is even). Curves are shown for $N = 2, \dots, 14$, with $N = 2$ being the strongest bound.

The idea of increasing $\dim(H^\dagger H)$ to ameliorate the hierarchy problem is an old one. In traditional Technicolor models, the role of the Higgs is played by a fermion condensate $\bar{\psi}\psi$ with dimension 3, so that the ‘mass’ term $(\bar{\psi}\psi)^2$ is irrelevant. A basic tension in this setup is that the ‘Yukawa’ terms $(\bar{\psi}\psi)\bar{q}u$ which generate fermion masses after EWSB are also irrelevant. To correctly account for the top-mass, we must imagine that such terms are suppressed by a low scale in the Lagrangian $\mathcal{L}_{\text{Yuk.}} \supset \frac{1}{\Lambda_{\text{low}}^2}(\bar{\psi}\psi)\bar{q}u$. But this same low scale would then generically appear in other four-fermion operators, leading to dangerous flavor-changing neutral currents.

Conformal Technicolor (CTC) [8] seeks to avoid this tension by assuming that H participates in strong conformal dynamics above the electroweak scale, which generates a large dimension

for $H^\dagger H$, while the dimension of H remains near 1. While this idea is intriguing, we will show that it needs additional assumptions to work in practice. In particular, our bounds definitively rule out the simplest ‘flavor-generic’ CTC models.

To begin, let us determine the range of $d = \dim(H)$ and $\Delta = \dim(H^\dagger H)$ that is phenomenologically viable in CTC. Firstly, we must require that y_t remain perturbative throughout the conformal regime, which places an upper bound on the possible running distance. Indeed, suppose conformal dynamics occurs between $\Lambda_{\text{EW}} \approx 4\pi v$ and some higher scale Λ_{UV} . Within this range of energies, Yukawa couplings run according to

$$y_i(\mu) = \left(\frac{\mu}{\Lambda_{\text{EW}}} \right)^{d-1} y_i(\Lambda_{\text{EW}}) \quad (4.62)$$

(ignoring corrections from small perturbations away from exact conformal symmetry, like SM gauge couplings and other Yukawa couplings). Requiring $y_t \lesssim 4\pi$ for all $\mu \in [\Lambda_{\text{EW}}, \Lambda_{\text{UV}}]$ then gives

$$\frac{\Lambda_{\text{UV}}}{\Lambda_{\text{EW}}} \lesssim \left(\frac{\Lambda_{\text{EW}}}{m_t} \right)^{\frac{1}{d-1}}. \quad (4.63)$$

Secondly, we must ensure that small perturbations of the theory by the Higgs mass operator $H^\dagger H$ don’t destabilize the conformal dynamics. This is certainly the case if $H^\dagger H$ is irrelevant, $\Delta \geq 4$. If on the other hand $\Delta < 4$, then we must also impose the lower bound,

$$\frac{\Lambda_{\text{UV}}}{\Lambda_{\text{EW}}} \lesssim \left(\frac{1}{c(\Lambda_{\text{UV}})} \right)^{\frac{1}{4-\Delta}}, \quad (4.64)$$

where $c(\Lambda_{\text{UV}})$ is the coefficient of $H^\dagger H$ in the perturbation $\delta\mathcal{L} = c(\Lambda_{\text{UV}})H^\dagger H$ at Λ_{UV} . The strength of the bound Eq. (4.64) varies, depending on the amount of tuning we’re willing to tolerate in this coefficient.

Finally, while Eqs. (4.63) and (4.64) prefer a small running distance, Λ_{UV} must also be sufficiently large to suppress problematic flavor-changing operators, such as $(ds^c)(\overline{s}d^c)$ which contributes to K - \overline{K} mixing. In a ‘flavor-generic’ model, we should demand

$$\frac{1}{\Lambda_{\text{UV}}^2} \lesssim \frac{1}{\Lambda_F^2} \quad (\text{generically}), \quad (4.65)$$

where $\Lambda_F \sim 3.2 \times 10^5$ TeV for CP-violating contributions to K - \bar{K} mixing [111]. More optimistically, we might imagine that $(ds^c)(\bar{s}d^c)$ is generated with Yukawa suppression, so that the constraint above gets modified to

$$\frac{y_d(\Lambda_{UV})y_s(\Lambda_{UV})}{\Lambda_{UV}^2} \lesssim \frac{1}{\Lambda_F^2} \quad (\text{optimistically}), \quad (4.66)$$

with $y_i(\Lambda_{UV})$ given by Eq. (4.62).

Together, these requirements restrict viable models to a particular region of the d - Δ plane, which can then be compared with our bounds. In models where the conformal dynamics is custodially-symmetric, H transforms as a fundamental of $\text{SO}(4) \cong \text{SU}(2)_L \times \text{SU}(2)_R$ (which is weakly gauged by SM gauge fields). However, the assumption of custodial symmetry is not actually necessary for us because our bound for $\text{SU}(2)_L$ alone is identical to our bound for $\text{SO}(4)$.

The viable regions for flavor-generic and flavor-optimistic CTC models are shown in figure 4.5, superimposed with our strongest $\text{SU}(2)$ dimension bound. The right-hand edge of the viable regions comes from the combination of Eq. (4.65) with Eq. (4.63), while the bottom edges come from the combination of Eq. (4.65) with Eq. (4.64) for different values of $c(\Lambda_{UV})$. We see that for reasonable assumptions about the coefficient $c(\Lambda_{UV})$, flavor-generic models are ruled out. This conclusion remains true even if the conformal dynamics respects CP symmetry, in which case the effective flavor scale can be closer to $\Lambda_F \sim 10^4$ TeV.

By contrast, flavor-optimistic models with reasonable tunings $c(\Lambda_{UV}) \lesssim 0.1$ and somewhat large dimensions $d \sim 1.3$ - 1.5 are not necessarily ruled out. Our bound does place an upper limit on the scale of new physics Λ_{UV} , but with sufficient Yukawa suppression these upper limits can be phenomenologically acceptable. For instance, with $c = 0.01$, Λ_{UV} must lie below 6.8×10^3 TeV, while $c = 0.1$ gives $\Lambda_{UV} \lesssim 1.6 \times 10^3$ TeV. At some point however, the predictions for these models become essentially those of minimal flavor violation with a low flavor scale, and strong conformal dynamics seems more and more like a gratuitous assumption.

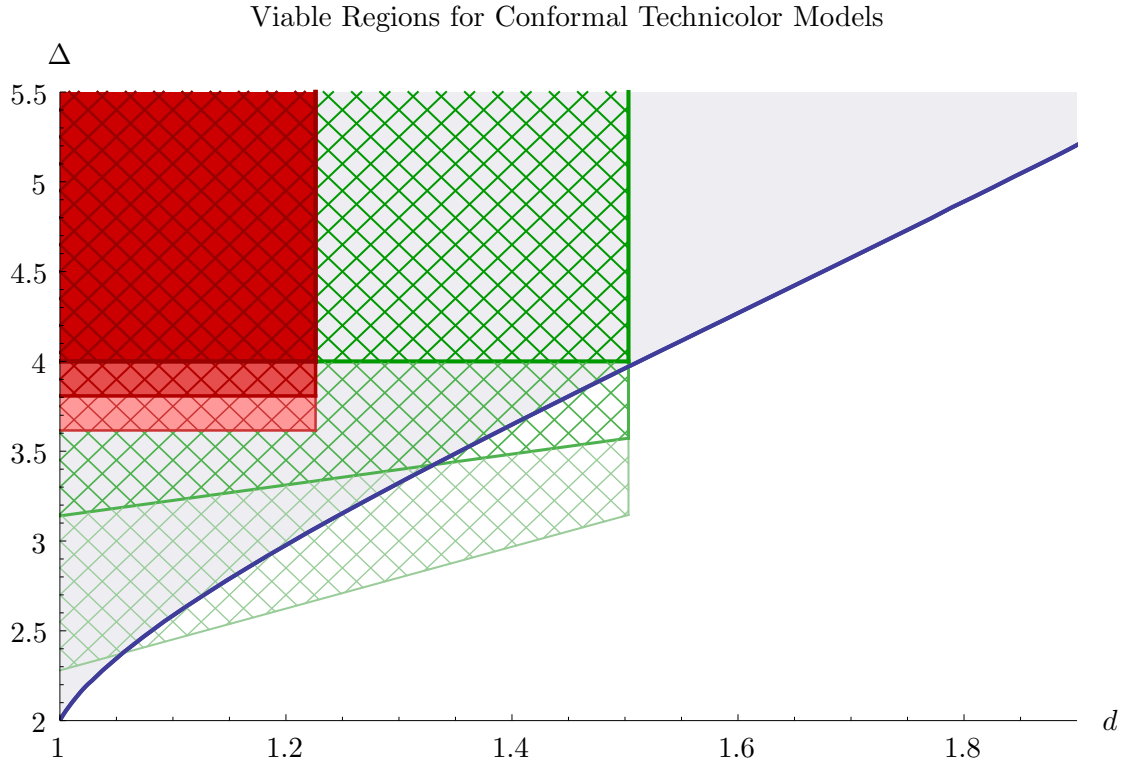


Figure 4.5: Viable regions for conformal technicolor models in the flavor-generic (red) and flavor-optimistic (cross-hatched green) cases are shown superimposed with our bound (blue, excluding the gray-shaded region). Regions for $c(\Lambda_{UV}) = 1, 0.1$, and 0.01 are shown in successively lighter shades of each color, with the largest region corresponding to $c(\Lambda_{UV}) = 0.01$ in each case. Flavor-generic models are ruled out.

4.3.3 Symmetric Tensors in $SO(N)$ Theories

It is straightforward to modify our procedure to obtain bounds on symmetric tensors $\mathcal{O}_{(ij)}$ appearing in $\phi_i \times \phi_j$. To bound a symmetric tensor with dimension Δ_0 and spin ℓ_0 , we look for a linear functional satisfying the normalization condition

$$\alpha \begin{pmatrix} F_{\Delta_0, \ell_0} \\ (1 - \frac{2}{N}) F_{\Delta_0, \ell_0} \\ -(1 + \frac{2}{N}) H_{\Delta_0, \ell_0} \end{pmatrix} = 1, \quad (4.67)$$

as well as $\alpha(V) \geq 0$ for all other vectors V in the $SO(N)$ sum rule.

Figure 4.6 shows the resulting dimension bound on $\phi_{(i}\phi_{j)}$ (the lowest dimension scalar symmetric tensor appearing in $\phi_i \times \phi_j$) in the case of $\text{SO}(4)$ symmetry. Note that this bound does not apply in a simple way to operators in theories with $\text{SU}(2)$ symmetries, because there is no coincidence between $\text{SU}(N)$ and $\text{SO}(2N)$ bounds for non-singlets.

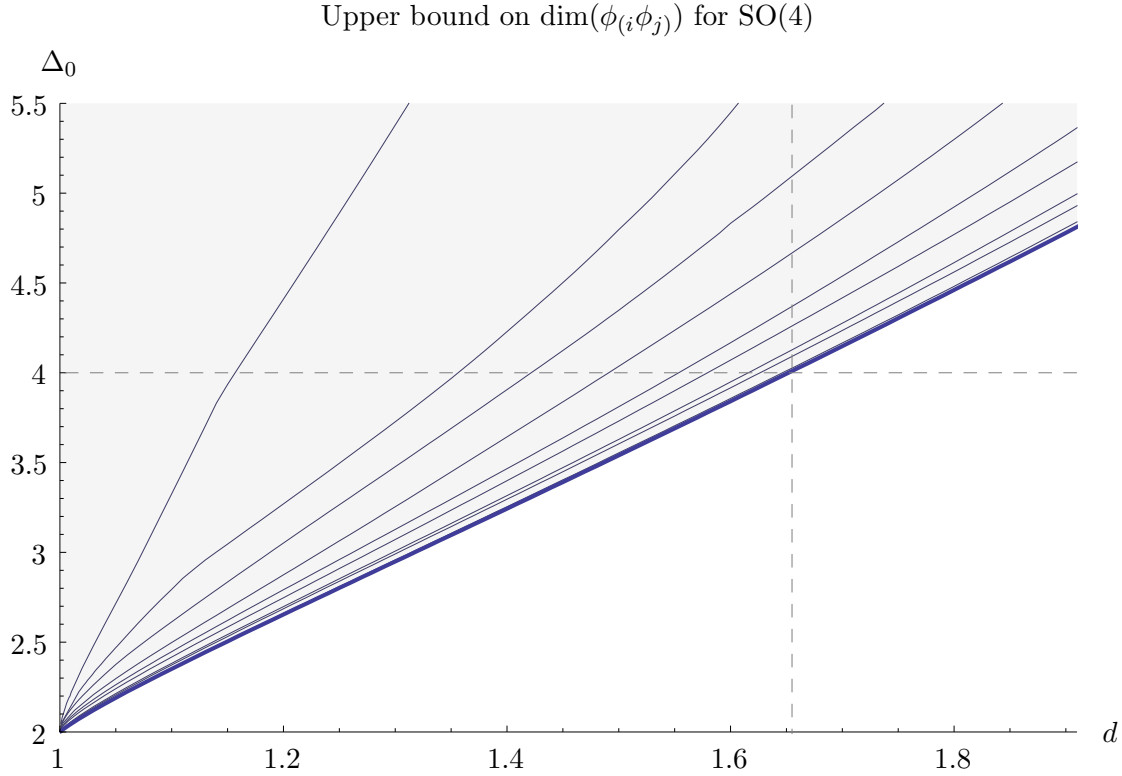


Figure 4.6: An upper bound on the lowest dimension symmetric tensor scalar appearing in $\phi \times \phi$, where ϕ transforms in the fundamental of $\text{SO}(4)$. Here we show $k = 2, \dots, 11$.

4.3.4 Superconformal Theories

Now let us turn to bounding operator dimensions in superconformal theories, using the sum rule Eq. (4.31). A bound on $\dim(\Phi^\dagger \Phi)$ in terms of $\dim(\Phi)$ was first obtained in [57] using only the middle row of Eq. (4.31). In [60], it was shown that the bound could be improved by incorporating the other rows, and linear programming calculations were given up to $k = 4$. In

figure 4.7, we present a new version of these bounds for k up to 11, corresponding to a 198-dimensional search space.

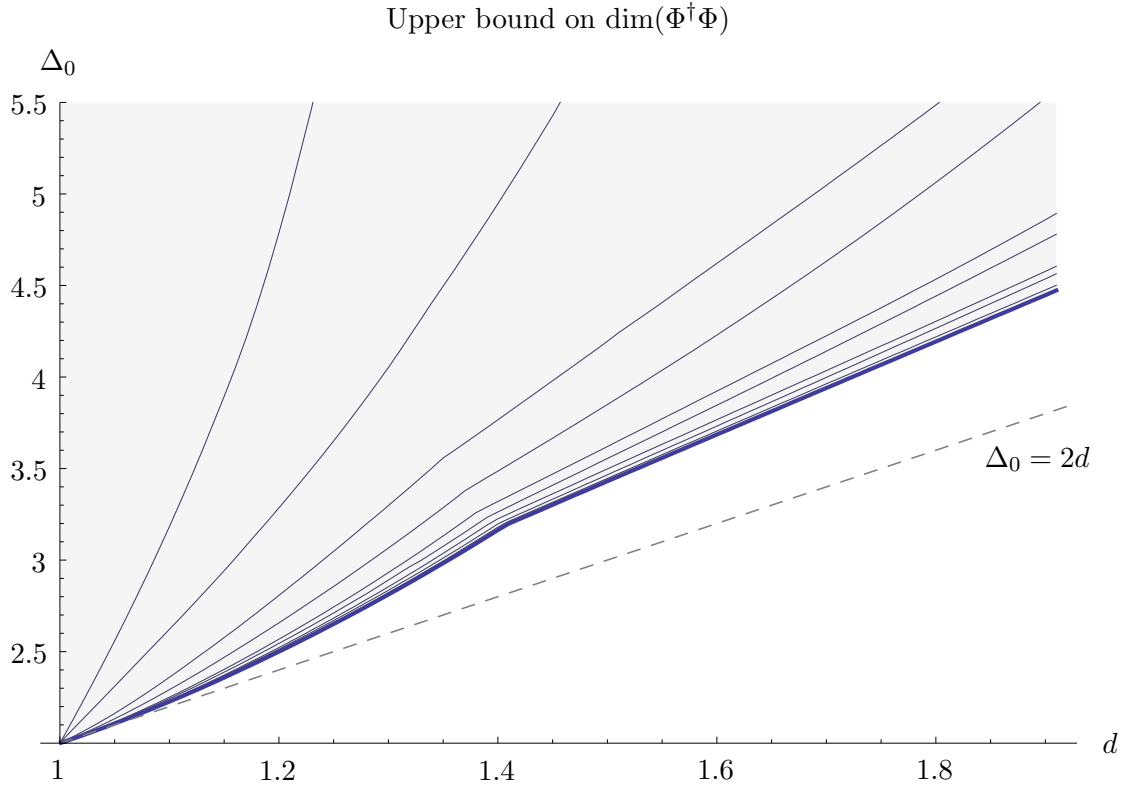


Figure 4.7: An upper bound on the dimension of $\Phi^\dagger \Phi$, where Φ is a chiral primary scalar of dimension d . The dashed line is the factorization value $\Delta = 2d$. Here we show $k = 2, \dots, 11$.

Several interesting new features emerge at large k . Most strikingly, the bound appears to be tangent to the factorization line $\Delta_0 = 2d$ near $d = 1$. Figure 4.8 shows a higher-resolution plot for small values of d , which displays this behavior more clearly. An approximate fit to the $k = 11$ curve in figure 4.8 is given by

$$\Delta_0 \leq 2(1 + \epsilon) + 2.683 \epsilon^2 + \dots \quad (\epsilon \ll 1), \quad (4.68)$$

where $d = 1 + \epsilon$. Note that known superconformal theories populate the entire factorization line,¹⁰

¹⁰Namely supersymmetric mean field theories, which satisfy the necessary requirements of unitarity and crossing symmetry, and exist for each $d \geq 1$. They occur in the infinite- N limit of supersymmetric gauge theories.

so it is impossible to have a bound stronger than $\Delta_0 \leq 2d$. Our bound on $\dim(\Phi^\dagger\Phi)$ is one of the few examples computed to date that approaches the provably best possible bound for some nontrivial range of d 's.

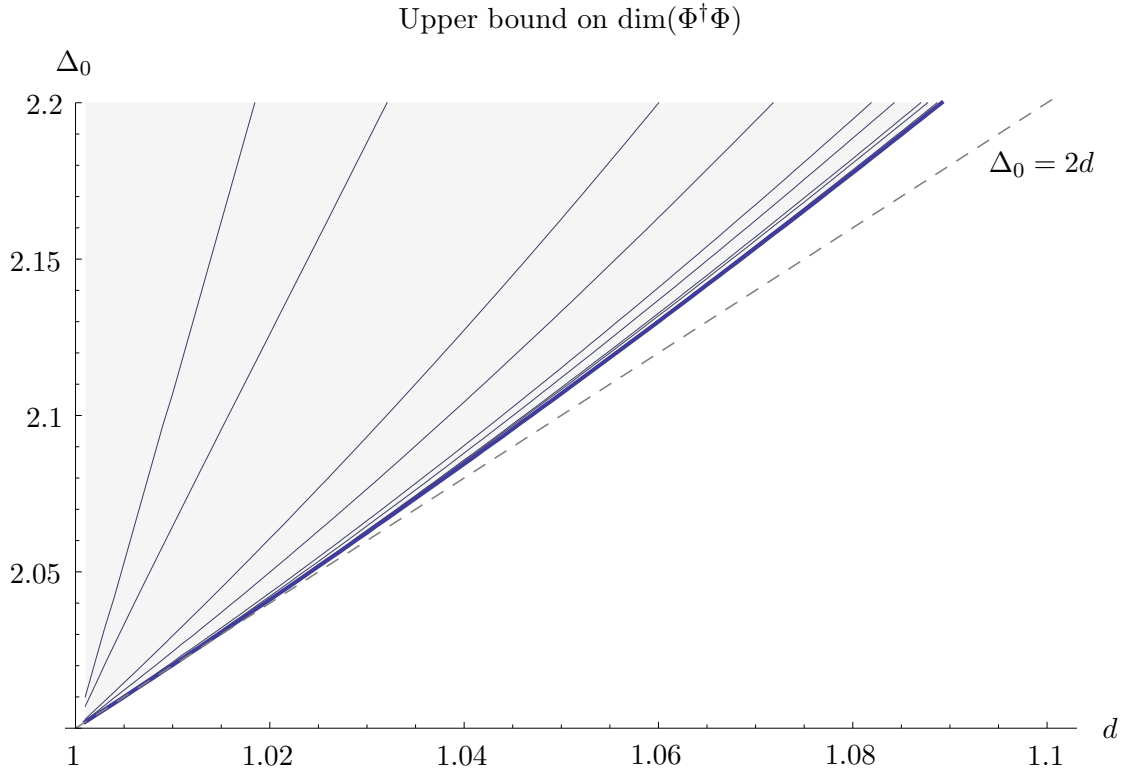


Figure 4.8: A zoom in on the region of figure 4.7 near $\dim(\Phi) = 1$.

Eq. (4.68) can be directly tested in theories that admit a perturbative Banks-Zaks limit and contain a chiral operator with dimension near 1. As far as we are aware, there are no known examples of perturbative theories living above the factorization line. Here we have shown numerically that this can be understood purely from the constraints of crossing symmetry and unitarity. It would be very interesting to understand this fact analytically.

It is amusing to speculate on the form of the bound as $k \rightarrow \infty$. A simple and intriguing possibility is that the small- d behavior might extend to all d , so that the best possible bound

$\Delta_0 \leq 2d$ is realized. In other words, it might be the case that the anomalous dimension $\gamma_{\Phi^\dagger\Phi} = \dim(\Phi^\dagger\Phi) - 2\dim(\Phi)$ is always non-positive. This possibility was investigated recently for theories with a weakly-coupled gravity dual in [102], with inconclusive results; effective field theories in AdS_5 allow for both positive and negative contributions to $\gamma_{\Phi^\dagger\Phi}$. However, it's possible that additional constraints might be present in those theories which admit a consistent UV completion.

Another possibility is that the bound converges above the factorization line, with a shape similar to the $k = 11$ curve in figure 4.7. In that case, one might wonder about the significance of the cusp near $d = 1.4$, which appears to be a common feature of each curve with $k \geq 4$. A previous example of a dimension bound with a cusp is the 2D real scalar dimension bound, presented in [55] (building on the first 2D results of [54]). There, an actual theory, the 2D Ising model, exists very near the cusp, so that the bound is close to the best possible at that value of d . By analogy, one might speculate that an $\mathcal{N} = 1$ SUSY ‘minimal model’ exists in the cusp in figure 4.7.

Phenomenological Applications

Our bound on $\dim(\Phi^\dagger\Phi)$ has implications for several models that use strong superconformal dynamics to tailor soft parameters in the MSSM. One example is the solution to the $\mu/B\mu$ problem in gauge mediation proposed in [32, 33] and further developed in [34–37]. In this scenario, SUSY breaking is communicated to the visible sector via a chiral field X which develops a SUSY-breaking VEV $\langle X \rangle = F\theta^2$ at some scale Λ_{IR} . In matching to the MSSM at Λ_{IR} , the effective operators

$$\mathcal{O}_X = c_X \int d^4\theta \frac{X^\dagger H_u H_d}{M_*} + \text{h.c.} \quad \text{and} \quad \mathcal{O}_{X^\dagger X} = c_{X^\dagger X} \int d^4\theta \frac{X^\dagger X H_u H_d}{M_*^2}, \quad (4.69)$$

contribute to μ and $B\mu$, respectively. Here, M_* is the scale where these operators originate (typically the messenger scale). Many of the simplest gauge-mediated models generate both \mathcal{O}_X and $\mathcal{O}_{X^\dagger X}$ at one-loop at the messenger scale, so that naïvely $c_X \sim c_{X^\dagger X} \sim \frac{\lambda^2}{16\pi^2}$, with λ an $O(1)$ coupling

constant. However, this then leads to the problematic relation $B\mu/\mu^2 \sim 16\pi^2$, which precludes viable electroweak symmetry breaking.

The solution proposed in [32, 33] is that X should participate in strong conformal dynamics over some range of scales $\Lambda_{\text{IR}} < \mu < \Lambda_{\text{UV}}$, with $\Lambda_{\text{UV}} \leq M_*$. If the anomalous dimension $\gamma_{X^\dagger X} \equiv \dim(X^\dagger X) - 2\dim(X)$ is positive, then the operator $\mathcal{O}_{X^\dagger X}$ will be suppressed relative to \mathcal{O}_X , and $B\mu/\mu^2$ can be close to unity at the matching scale Λ_{IR} . In particular, to restore proper electroweak symmetry breaking, we should approximately have

$$\left(\frac{\Lambda_{\text{IR}}}{\Lambda_{\text{UV}}}\right)^{\gamma_{X^\dagger X}} \approx \frac{1}{16\pi^2}. \quad (4.70)$$

Using this relation, our upper bound on $\dim(X^\dagger X)$ in figure 4.7 translates into a lower bound on the running distance $\Lambda_{\text{UV}}/\Lambda_{\text{IR}}$, shown in figure 4.9. Note in particular that a small $\dim(X)$ requires a very large running distance, since our bound on $\gamma_{X^\dagger X}$ approaches zero as $\dim(X) \rightarrow 1$. Consequently, viable models should at least have $\dim(X) \gtrsim 1.3$. Note that $\dim(X)$ can almost always be calculated using a -maximization in concrete examples, so a bound on the required running distance can be easily read from figure 4.9 for specific models.

Our bound can also apply to models of conformal sequestering [20, 21, 23–28, 33] which contain chiral gauge singlets, where the idea is that a large $\dim(X^\dagger X)$ can lead to suppression of flavor-dependent soft-mass operators,

$$c_{ij} \int d^4\theta \frac{1}{M_*^2} X^\dagger X \phi_i^\dagger \phi_j. \quad (4.71)$$

Let us for example assume a gravity mediated scenario, where the cutoff scale is $M_* \sim M_{\text{pl}}$ and conformal running occurs between M_{pl} and an intermediate scale $\Lambda_{\text{int}} \sim 10^{11}$ GeV. Viable flavor physics then roughly requires $\dim(X^\dagger X) - 2 \gtrsim 1$ [27], and from figure 4.7 we see that such models should also have $\dim(X) \gtrsim 1.35$ or so.¹¹ Our bounds similarly constrain the possible suppression of

¹¹However, it's possible that one could avoid these constraints by having 'safe' flavor currents appear in the OPE (as discussed in [27]).

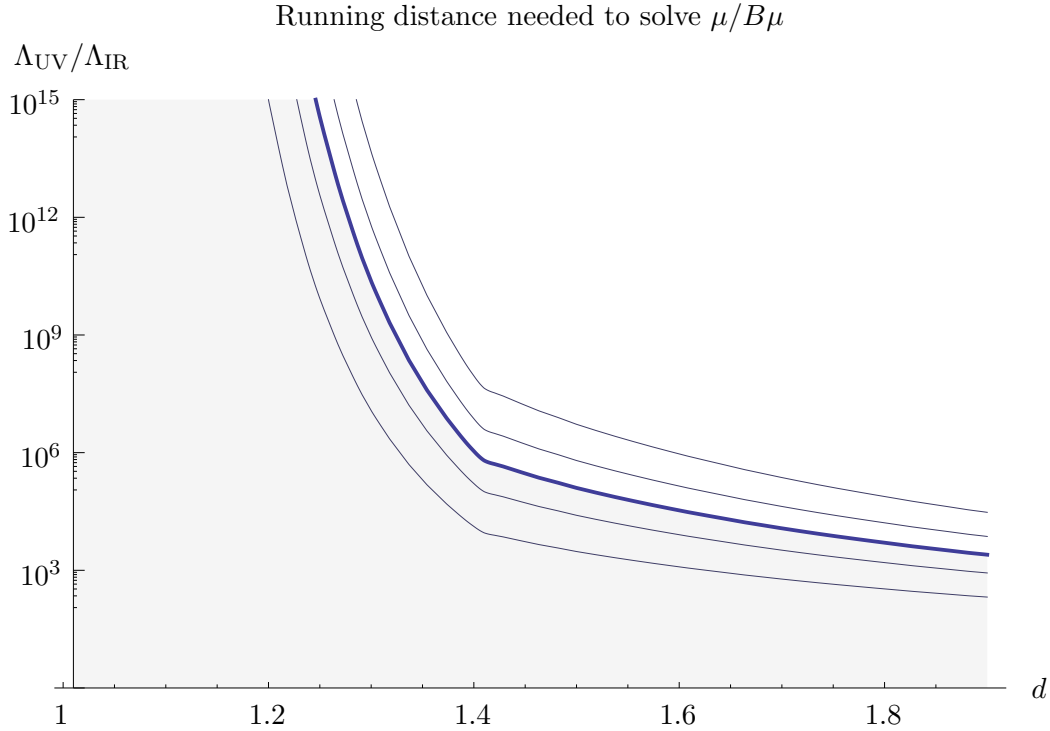


Figure 4.9: An approximate lower bound on the running distance required for solving the $\mu/B\mu$ problem with strong conformal dynamics, as a function of $d = \dim(X)$. The middle curve corresponds to a loop factor suppression: $c_{X^\dagger X}(\Lambda_{\text{IR}}) = \frac{1}{16\pi^2} c_X^2(\Lambda_{\text{IR}})$, while the outer curves correspond to suppressions within factors of 2 and 5 of a loop factor.

these operators in superconformal flavor models [15–19, 22, 29–31], where the visible sector fields participate in the strong conformal dynamics. Once again, in all of these situations a comparison to our bounds can be checked in concrete examples using a -maximization.

4.4 Bounds on OPE Coefficients

In this section we will turn our attention away from bounding operator dimensions and instead explore some of the more basic bounds on OPE coefficients obtainable using these methods. We’ll begin by reproducing (and strengthening) the upper bounds on scalar OPE coefficients for general CFTs previously presented in [56]. Then we’ll focus on something qualitatively new —

the possibility of placing *lower* bounds on OPE coefficients in theories that have a gap in the spectrum of operator dimensions. In fact, this happens naturally in supersymmetric theories for protected operators appearing in the $\Phi \times \Phi$ OPE, where a gap is forced by unitarity. We will then demonstrate that there are extremely constraining upper and lower bounds on the OPE coefficients of these operators when $\dim(\Phi) < 3/2$.

4.4.1 Scalar Operators in General Theories

Let us begin by producing bounds on OPE coefficients of scalar operators \mathcal{O}_0 of dimension Δ_0 appearing in the $\phi \times \phi$ OPE, where ϕ is a scalar operator of dimension d . As we saw in Eq. (4.39), by applying a linear functional α to the CFT crossing relations we can obtain an upper bound $\lambda_{\mathcal{O}_0}^2 \leq -\alpha(F_{0,0})$. In figure 4.10 we show the best upper bounds on $\lambda_{\mathcal{O}_0}$ as a function of Δ_0 that we have obtained so far, for $d = 1.01, \dots, 1.66$ with a spacing of 0.05. These bounds are obtained using $k = 11$, corresponding to a 66-dimensional search space. This plot strengthens bounds previously presented in [56].

Figure 4.10 clearly contains a lot of interesting structure. First, as $d \rightarrow 1$, the curve becomes more and more sharply peaked around $\Delta_0 \simeq 2$, with the height of the peak converging to the free value $\lambda_0 = \sqrt{2}$.¹² On the other hand, as $\Delta_0 \rightarrow 1$ all of the curves drop sharply to zero (first peaking at larger values of d), corresponding to the fact that a free operator cannot appear in the OPE. All of the bounds also increase in strength as Δ_0 becomes large, possibly asymptoting to zero. Finally, as d increases at fixed Δ_0 the bounds monotonically decrease in strength. Note that in the present study we have found the region $d > 1.66$ to be numerically more difficult (though very weak bounds appear to exist at least up to $d \sim 1.86$), and we postpone a full investigation of this region to future work.

¹²Note that the free OPE coefficient is $\sqrt{2}$ rather than 1 because we have required the ϕ^2 operator to have a canonically normalized two-point function, rather than the normalization inherited from Wick contractions.

Let us take a moment to understand a way in which our method fails to fully pick out the spectrum of free theories as $d \rightarrow 1$. While our upper bound becomes nicely peaked around the free value in this limit, our algorithm cannot easily distinguish between a single $\Delta_0 \simeq 2$ operator with $\lambda_0 \simeq \sqrt{2}$, and a broader spectrum of operators, each having Δ_0 somewhat close to 2 and $\lambda_0 < \sqrt{2}$. The issue is that both of these scenarios can lead to very similar conformal block contributions to the 4-point functions that we are studying. On the other hand, if we knew that there was only a *single* operator appearing in the OPE up to a certain dimension, this ambiguity could not occur and we would be able to also place lower bounds on its OPE coefficient. In the next subsection we will study this possibility in more detail, focusing on protected operators appearing in the $\Phi \times \Phi$ OPE in SCFTs.

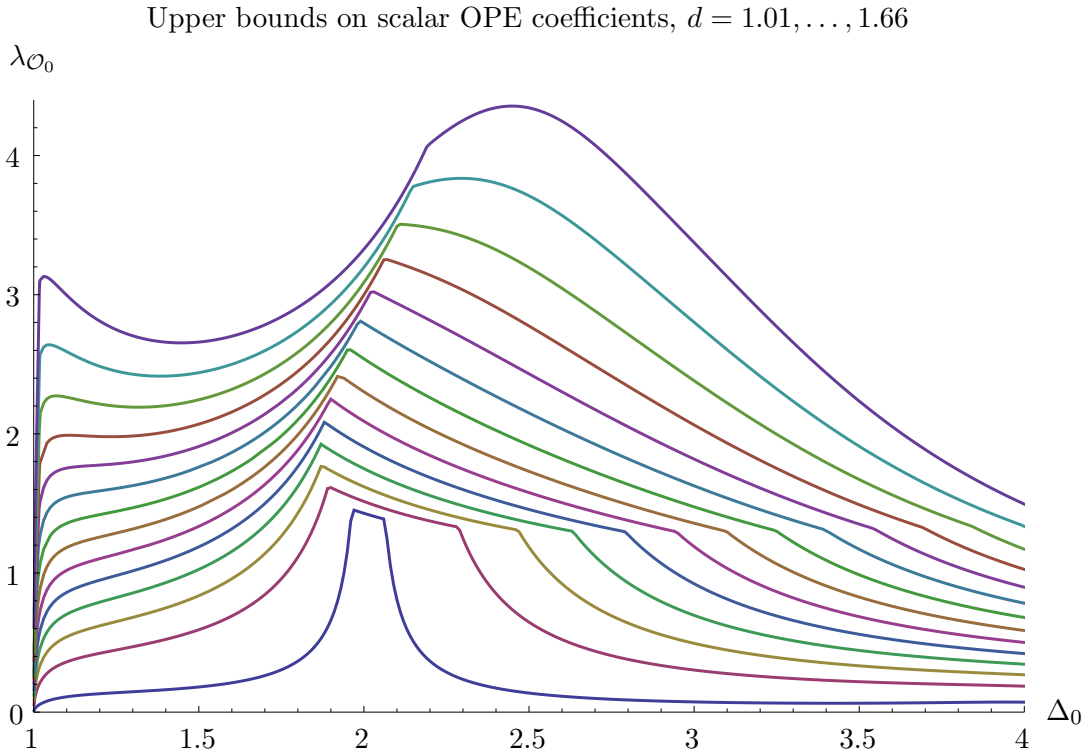


Figure 4.10: Upper bounds on the OPE coefficient of a scalar operator $\mathcal{O}_0 \in \phi \times \phi$ (not necessarily of lowest dimension). Each curve is for a different value $d = 1.01, \dots, 1.66$, with a spacing of 0.05 and $d = 1.01$ corresponding to the lowest curve. Here we have taken $k = 11$.

4.4.2 Protected Operators in Superconformal Theories

As we reviewed in section 4.2.3, if Φ is a chiral superconformal primary of dimension d in an $\mathcal{N} = 1$ SCFT, the $\Phi \times \Phi^\dagger$ OPE contains superconformal primaries of dimension $\Delta \geq \ell + 2$ and their descendants. On the other hand, the $\Phi \times \Phi$ OPE can contain a chiral Φ^2 operator of dimension $2d$, superconformal descendants $\overline{Q}\mathcal{O}_\ell$ of protected operators having dimension $2d + \ell$, and superconformal descendants $\overline{Q}^2\mathcal{O}$ of unprotected operators with a dimension satisfying $\Delta \geq |2d - 3| + 3 + \ell$.

Notice that, as long as $d < 3/2$, there is necessarily a gap between the dimensions of the protected operators appearing in the $\Phi \times \Phi$ OPE and the dimensions of the unprotected operators. This gap is a consequence of the unitarity constraints on operator dimensions in SCFTs. Because of this gap, no other operators appearing in the OPE can give similar conformal block contributions to the four-point function $\langle \Phi \Phi^\dagger \Phi \Phi^\dagger \rangle$, and we can attempt to derive *lower* bounds on the OPE coefficients λ_{Φ^2} and $\lambda_{\overline{Q}\mathcal{O}_\ell}$, in addition to upper bounds.

The logic used to obtain a lower bound requires only a slight modification to the procedure described in section 4.2. Since one could in principle attempt to obtain a lower bound in any theory with a dimension gap, let us first describe the logic for the simplest case of the real scalar crossing relation in general CFTs. To obtain a lower bound on an OPE coefficient $\lambda_{\mathcal{O}_0}^2$, we can again consider applying a linear functional to the real scalar crossing relation, as in Eq. (4.36). However, instead of imposing the constraints (4.37) and (4.38), we can alternatively require

$$\alpha(F_{\Delta_0, \ell_0}) = 1, \quad \text{and} \quad (4.72)$$

$$\alpha(F_{\Delta, \ell}) \leq 0, \quad \text{for all other operators in the spectrum,} \quad (4.73)$$

which leads to the lower bound

$$\lambda_{\mathcal{O}_0}^2 = -\alpha(F_{0,0}) - \sum_{\mathcal{O} \neq \mathcal{O}_0} \text{pos.} \times \text{neg.} \geq -\alpha(F_{0,0}). \quad (4.74)$$

Note that (4.72) and (4.73) are only compatible with each other if we know that there is a gap between Δ_0 and the Δ 's for all other operators in the spectrum.

Generalizing to the superconformal crossing relation of Eq. (4.31), if we isolate a protected operator \mathcal{O}_0 of spin ℓ_0 and require

$$\alpha \begin{pmatrix} 0 \\ F_{2d+\ell_0, \ell_0} \\ -H_{2d+\ell_0, \ell_0} \end{pmatrix} = 1, \quad (4.75)$$

$$\alpha \begin{pmatrix} 0 \\ F_{\Delta, \ell} \\ -H_{\Delta, \ell} \end{pmatrix} \leq 0, \quad \text{for all other operators in } \Phi \times \Phi, \text{ and} \quad (4.76)$$

$$\alpha \begin{pmatrix} \mathcal{F}_{\Delta, \ell} \\ \tilde{\mathcal{F}}_{\Delta, \ell} \\ \tilde{\mathcal{H}}_{\Delta, \ell} \end{pmatrix} \leq 0, \quad \text{for all (non-unit) operators in } \Phi \times \Phi^\dagger, \quad (4.77)$$

we obtain the lower bound

$$\lambda_{\mathcal{O}_0}^2 \geq -\alpha \begin{pmatrix} \mathcal{F}_{0,0} \\ \tilde{\mathcal{F}}_{0,0} \\ \tilde{\mathcal{H}}_{0,0} \end{pmatrix}. \quad (4.78)$$

Meanwhile, reversing the inequalities in (4.76) and (4.77) leads to an upper bound on $\lambda_{\mathcal{O}_0}^2$, following our usual logic.

In figure 4.11 we show the resulting upper and lower bounds on λ_{Φ^2} , where we have taken $k = 2, \dots, 11$ in the numerical optimization. We can see that the strongest bounds are extremely constraining when $d = \dim(\Phi)$ is even somewhat close to 1, forcing λ_{Φ^2} to live very close to the free value $\lambda_{\Phi^2} = \sqrt{2}$. In particular, these results imply that it should not be possible to construct a weakly-coupled (Banks-Zaks) SCFT where both d and λ_{Φ^2} are modified at the one-loop

level. Indeed, in all constructible examples λ_{Φ^2} receives its leading correction at second order in perturbation theory. On the other hand, we see that the lower bound disappears before $d = 3/2$, as expected, while the upper bound persists.

As $d \rightarrow 2$, we may also compare the upper bound to the OPE coefficients of composite operators in theories containing free chiral superfields. In the simplest case, we can consider a single free field Q and then identify $\Phi \equiv \frac{1}{\sqrt{2}}Q^2$. In this case the operator $\Phi^2 \equiv \frac{1}{\sqrt{4!}}Q^4$ is canonically normalized, so the OPE is

$$\left(\frac{1}{\sqrt{2}}Q^2\right) \times \left(\frac{1}{\sqrt{2}}Q^2\right) \sim \frac{\sqrt{4!}}{2} \left(\frac{1}{\sqrt{4!}}Q^4\right) + \dots, \quad (4.79)$$

and we have $\lambda_{\Phi^2} = \sqrt{6}$, which is consistent with the bound. More generally, considering the dimension- n operator $\Phi \equiv \frac{1}{\sqrt{n!}}Q^n$ leads to an OPE coefficient of $\lambda_{\Phi^2} = \frac{(2n)!^{1/2}}{n!}$, which the bound must respect at even higher integer values of d .

Another simple generalization is to consider meson operators $M \equiv \frac{1}{\sqrt{2N}}Q^i Q_i$ built out of N free quarks Q^i . In this case Wick contractions give a two-point function $\langle (M^2)(M^2)^\dagger \rangle \sim 2 + \frac{4}{N}$, so the OPE in terms of canonically normalized operators is given by

$$M \times M \sim \sqrt{2 + \frac{4}{N}} \left(\frac{1}{\sqrt{2 + \frac{4}{N}}} M^2 \right) + \dots \quad (4.80)$$

Thus, we can read off an OPE coefficient of $\lambda_{\Phi^2} = \sqrt{2 + 4/N}$, which is consistent with our bound for all values of N . It is interesting to see that while OPE coefficients of composite operators with $d \sim 2$ know about the underlying constituents of the operator, as $d \rightarrow 1$ the OPE coefficient necessarily loses memory of where the operator came from. Indeed, free operators have no hair!

In figure 4.12 we extend these upper and lower bounds to OPE coefficients of the other protected operators $\overline{Q}\mathcal{O}_\ell$ appearing in the $\Phi \times \Phi$ OPE. Here we give the results for $\ell = 2, \dots, 10$ and have taken $k = 11$ in the numerical optimization (though similar bounds also exist at larger values of ℓ). All of the bounds continuously interpolate to the free values as $d \rightarrow 1$, given by

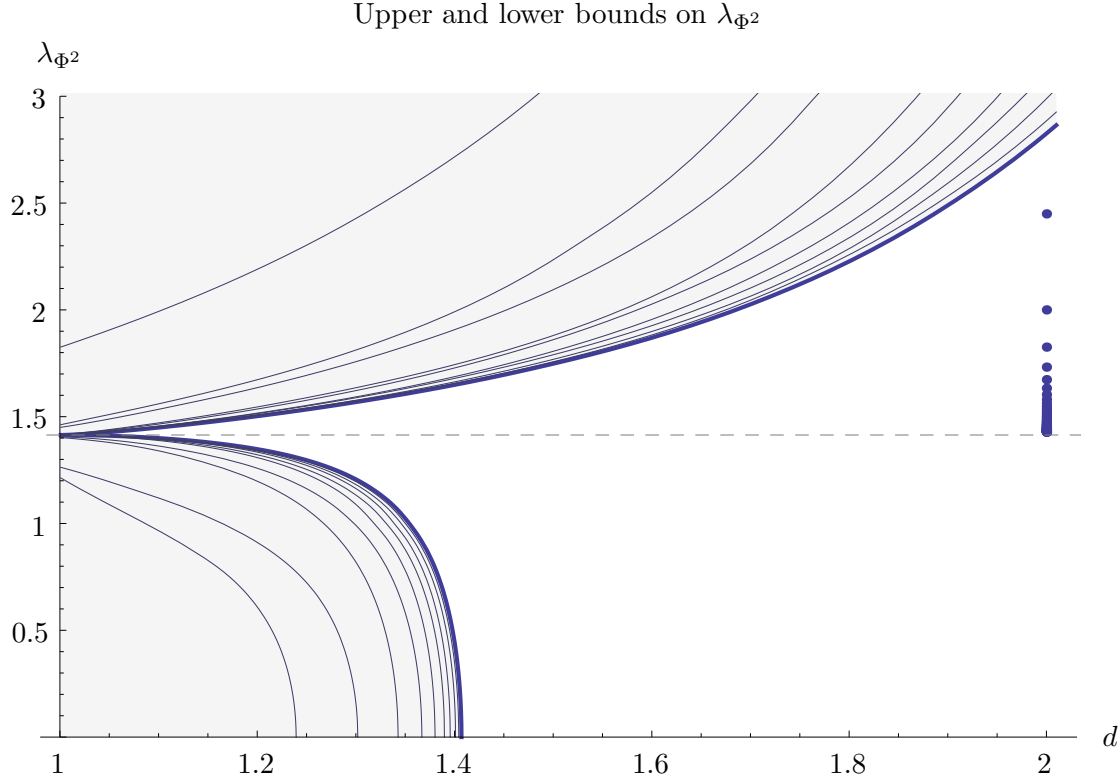


Figure 4.11: Upper and lower bounds on the OPE coefficient of Φ^2 in $\Phi \times \Phi$, as a function of $d = \dim(\Phi)$. The dashed line indicates the free value $\lambda_{\Phi^2} = \sqrt{2}$. The points shown at $d = 2$ indicate the sequence of values $\lambda_{\Phi^2} = \sqrt{2 + \frac{4}{N}}$ realized for composite operators in free theories. We give the bounds for $k = 2, \dots, 11$.

$\lambda_{\overline{Q}\mathcal{O}_\ell} = \sqrt{2} \frac{\ell!}{(2\ell)!^{1/2}}$. Notice that all lower bounds vanish before $d = 3/2$, as they should.¹³

Taken together, the upper and lower bounds on $\lambda_{\overline{Q}\mathcal{O}_\ell}$ are extremely strong, almost determining this coefficient when $d \lesssim 1.4$. One can view this singling out of an essentially unique OPE coefficient as a remarkable success of the 4D conformal bootstrap program! It is worth comparing the bounds to the known values of $\lambda_{\overline{Q}\mathcal{O}_\ell}$ in supersymmetric mean field theories (MFTs), which occur in the planar limit of large- N gauge theories. There, the role of $\overline{Q}\mathcal{O}_\ell$ is played by the ‘twist-2d’

¹³Once they are computed, one can include information about these lower bounds in semidefinite programs for other quantities, like e.g. upper bounds on OPE coefficients of operators in the $\Phi \times \Phi^\dagger$ OPE. We found that this procedure does not significantly improve the results in practice.

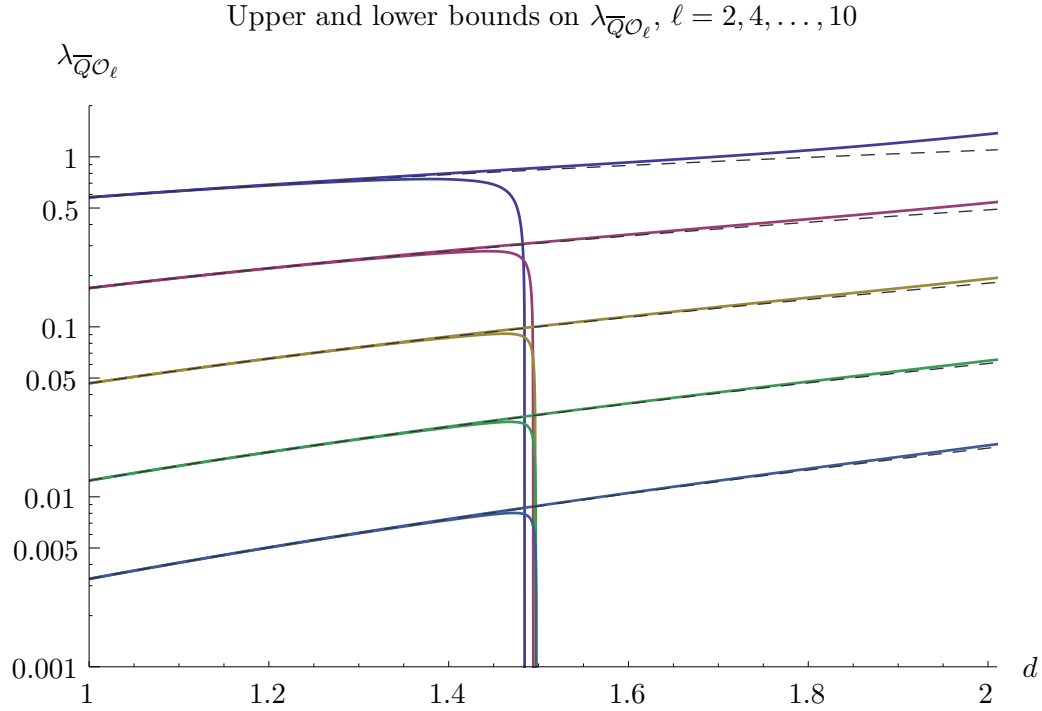


Figure 4.12: Upper and lower bounds on the OPE coefficients of protected operators $\overline{\mathcal{Q}}\mathcal{O}_\ell$ appearing in $\Phi \times \Phi$, along with their mean field theory values Eq. (4.82) (dashed lines), for $\ell = 2, 4, \dots, 10$. Each curve goes continuously to the free value $\sqrt{2} \frac{\ell!}{(2\ell)!^{1/2}}$ at $d = 1$. All lower bounds vanish at $d = 3/2$, since the gap in dimensions between $\overline{\mathcal{Q}}\mathcal{O}_\ell$ and non-protected operators disappears at that point. Here we have taken $k = 11$.

double-trace operators

$$\mathcal{O}_\ell^{(2)} \equiv \Phi \overleftrightarrow{\partial}^{\mu_1} \dots \overleftrightarrow{\partial}^{\mu_\ell} \Phi - \text{traces}, \quad (4.81)$$

with even spin ℓ . Their (squared) OPE coefficients in $\Phi \times \Phi$ are given by [83]

$$\lambda_{\mathcal{O}_\ell^{(2)}}^2 = \frac{2\Gamma^2(d+\ell)\Gamma(2d+\ell-1)}{\Gamma^2(d)\Gamma(\ell+1)\Gamma(2d+2\ell-1)}, \quad (4.82)$$

and these values of $\lambda_{\mathcal{O}_\ell^{(2)}}$ are shown as dashed lines in figure 4.12, for $\ell = 2, 4, \dots, 10$. They are fully consistent with both our upper and lower bounds on $\lambda_{\overline{\mathcal{Q}}\mathcal{O}_\ell}$. Note that the MFT value of $\lambda_{\mathcal{O}_0^{(2)}}$ is equal to the free value $\sqrt{2}$, so it is consistent with our bounds in figure 4.11.

The striking agreement between our bounds and the mean field theory values of OPE

coefficients at small d has interesting implications for SCFTs with weakly-coupled AdS_5 duals. In such theories, corrections to OPE coefficients away from their MFT values can be computed in perturbation theory using Witten diagrams. Our bounds imply that corrections to $\lambda_{\mathcal{O}_\ell^{(2)}}^2$ must vanish to very high order in $(d-1)$, particularly at large ℓ . If any corrections were nonzero at finite values of $(d-1)$, then we would obtain sharp bounds on bulk coupling constants. We defer further exploration of these interesting constraints to future work.

4.5 Bounds on Central Charges

In this section we explore bounds on the OPE coefficient appearing in front of the stress tensor $T^{\mu\nu}$, which is a conserved spin-2 operator of dimension 4 that must be present in any CFT. Since this OPE coefficient is fixed by a Ward identity in terms of the central charge c of the theory (defined as the coefficient appearing in the two-point function $\langle T^{\mu\nu} T^{\gamma\delta} \rangle \propto c$), we will ultimately be deriving bounds on c . Previously, lower bounds on the central charge in both general CFTs and SCFTs were explored in [57, 58, 60]. The main new results of this section will be to extend these analyses to situations with global symmetries, where we will show that there are bounds on the central charge that scale with the size of the global symmetry representation.

4.5.1 General Theories

Let us begin by establishing some notation. The stress tensor is typically normalized as

$$\langle T^{\mu\nu}(x) T^{\gamma\delta}(0) \rangle = \frac{40c}{\pi^4} \frac{I^{\mu\gamma}(x) I^{\nu\delta}(x)}{x^8}, \quad (4.83)$$

where $I^{\mu\gamma}(x) = \eta^{\mu\gamma} - 2 \frac{x^\mu x^\gamma}{x^2}$ and c is the central charge appearing in the trace anomaly, $\langle T^\mu_\mu \rangle = \frac{c}{16\pi^2} (\text{Weyl})^2 - \frac{a}{16\pi^2} (\text{Euler})$, when the theory is placed on a curved background. In this normalization a free scalar has $c_{\text{free}} = \frac{1}{120}$ and a free Weyl fermion has $c_{\text{free fermion}} = \frac{1}{40}$.

The stress tensor is the local current generating the dilatation charge, where in radial quantization $D = -\int d\Omega \hat{x}_\mu x_\nu T^{\mu\nu}$ (the integral is over a three-sphere surrounding the origin). Requiring the action $D\phi(0) = d\phi(0)$ then fixes the OPE to have the form $T^{\mu\nu}(x)\phi(0) \sim -\frac{2d}{3\pi^2 x^6} (x^\mu x^\nu - \frac{1}{4}\eta^{\mu\nu} x^2) \phi(0) + \dots$, which leads to the stress tensor conformal block contribution

$$x_{12}^{2d} x_{34}^{2d} \langle \phi \phi \phi \phi \rangle \sim \frac{d^2}{360c} g_{4,2} \quad (\text{general CFTs}). \quad (4.84)$$

Generalizing to the situation where ϕ^i transforms under an $\text{SO}(N)$ or $\text{SU}(N)$ global symmetry, the stress tensor appears as an S^+ operator in the sum rules given in Eqs. (4.14) and (4.23), again with OPE coefficient $\lambda_T^2 = \frac{d^2}{360c}$. Note that a free real scalar transforming as an $\text{SO}(N)$ fundamental or a complex scalar transforming as an $\text{SU}(N/2)$ fundamental gives a contribution of $N c_{\text{free}}$ to the central charge.

To begin, in figure 4.13 we show the bounds on c obtained by applying our semidefinite programming algorithm to the case of a single real scalar ϕ , where we show curves for $k = 2, \dots, 11$ in the numerical optimization. We see that for $k \geq 6$, the bounds smoothly approach the free value c_{free} as $d \rightarrow 1$. This is consistent with and improves upon the bounds on c previously presented in [57, 58]. Note that here we are only assuming that the dimensions of operators appearing in the $\phi \times \phi$ OPE satisfy the unitarity bound — one could also assume that ϕ is the lowest dimension scalar in the theory to obtain somewhat stronger bounds at larger values of d as was done in [58]. However, here we make only the minimal assumption to allow for a more straightforward comparison to our other bounds.

In figure 4.14 we show bounds on c in the presence of $\text{SO}(N)$ or $\text{SU}(N/2)$ global symmetries for $N = 2, \dots, 14$. Here we have taken $k = 11$. We see that the bounds also smoothly approach the free values as $d \rightarrow 1$, scaling linearly with N . This greatly improves upon the bounds derived in [57, 58] (and given in figure 4.13) for theories with global symmetries. The reason for the improvement is that here we have incorporated the constraints of crossing symmetry for *all* operators in the ϕ^i

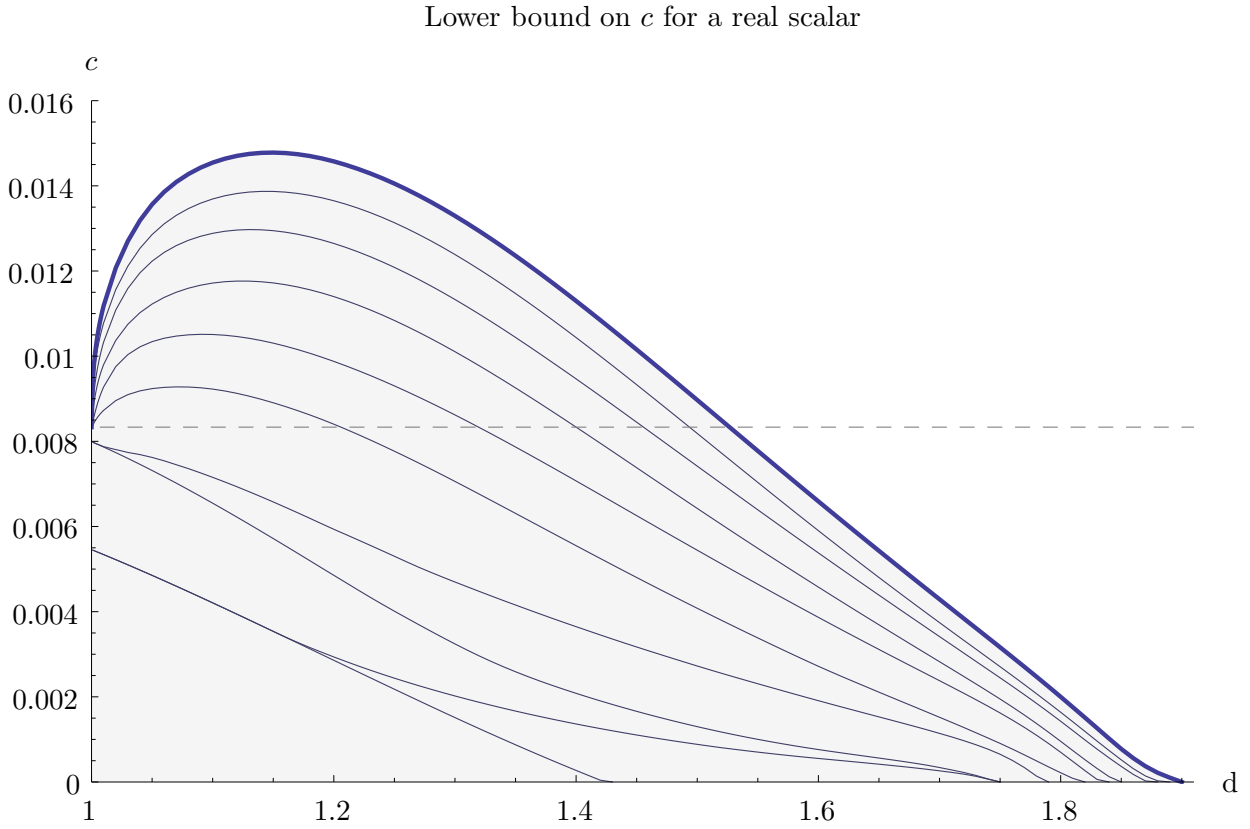


Figure 4.13: A lower bound on the central charge of a theory containing a scalar ϕ of dimension d . The dashed line indicates the value $c_{\text{free}} = 1/120$, corresponding to the central charge of a free scalar. Here we show bounds for the values $k = 2, \dots, 11$.

multiplet; without doing this the bounds of [57, 58] could not differentiate between the stress tensor and other spin 2 operators (e.g., the $\text{SO}(N)$ symmetric tensor $\phi^{(i} \partial^\mu \partial^\nu \phi^{j)}$) that have an $O(1)$ OPE coefficient in the $d \rightarrow 1$ limit.

It is interesting to understand the implications of the bound of figure 4.14 for the AdS/CFT correspondence. For theories with an AdS_5 dual description, the bulk Planck scale is proportional to c , the bulk gauge group is identified with the $\text{SO}(N)$ or $\text{SU}(N/2)$ global symmetry, and d is related to the masses of bulk fields. Our bound then says that theories with sufficiently light bulk excitations cannot have a gravitational scale that is arbitrarily small. Moreover, if those fields

transform as fundamentals under the bulk $\mathrm{SO}(N)$ or $\mathrm{SU}(N/2)$ gauge group (and correspond to operators with $d \sim 1$), then the Planck scale must scale at least linearly with N .

It would be fascinating to identify CFTs that live close to these bounds, particularly in the large N limit. Unfortunately, in gauge theories believed to flow to conformal fixed points that also possess an $\mathrm{SO}(N)$ or $\mathrm{SU}(N/2)$ global symmetry, the central charge typically scales as N^2 , at least near $d \sim 1$. The reason is that conformality forces the size of the global symmetry to scale proportionally to the size of the gauge group, and gauge degrees of freedom live in adjoint representations of the gauge group which have $O(N^2)$ components. We will see examples of this in the next subsection, where we extend the bounds to superconformal theories in which c is explicitly calculable.

4.5.2 Superconformal Theories

In $\mathcal{N} = 1$ SCFTs, the stress tensor is a superconformal descendant of the spin-1 $U(1)_R$ current, $T \sim (Q\bar{Q}J_R)_{\ell+1}$, as in Eq. (4.25). Applying Eq. (4.26) to (4.84), we see that J_R^μ has an OPE coefficient of $\lambda_R^2 = \frac{d^2}{72c}$, appearing as an S^+ operator in the superconformal sum rules of Eqs. (4.31) and (4.33). Since a free chiral superfield contains both a complex scalar and a Weyl fermion, it gives a contribution of $c_{\text{chiral}} = 2 \times \frac{1}{120} + \frac{1}{40} = \frac{1}{24}$.

In figure 4.15 we show the results of our semidefinite programming algorithm for obtaining bounds on the central charge of any theory containing a chiral scalar Φ . We give the results for $k = 2, \dots, 11$, where all of the curves for $k > 3$ drop sharply very close to $d \sim 1$ and go just below the free value. The $k = 11$ curve significantly improves upon SCFT central charge bounds previously obtained in [57, 60]. Note that the sharpness of the drop (reaching within 1% of the free chiral value closer than $d \sim 1.0000002$) is strong evidence that the free theory is an isolated solution to the crossing relations. This is intuitive from the perspective of constructing perturbations of the free theory — all such perturbations leading to an interacting SCFT require additional matter,

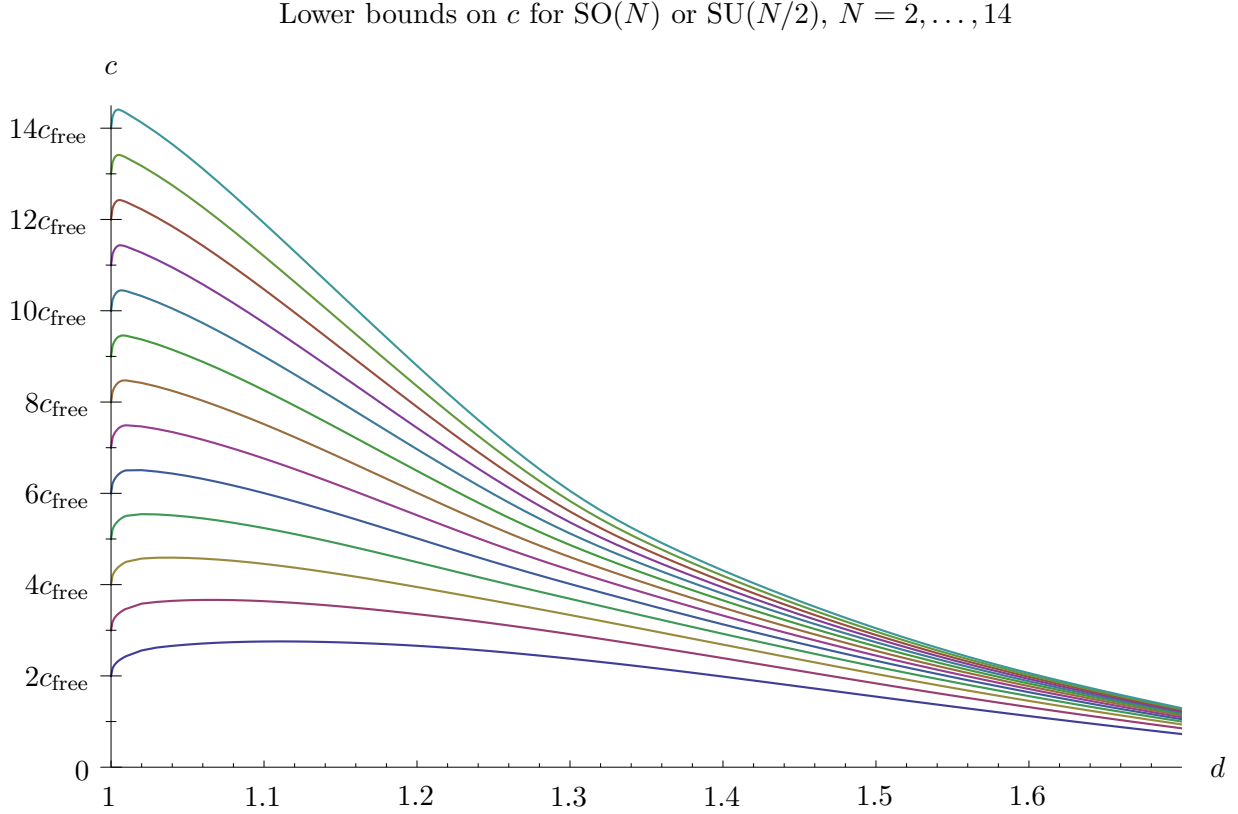


Figure 4.14: A lower bound on the central charge of a theory containing a scalar ϕ_i of dimension d transforming as a fundamental of an $\text{SO}(N)$ or $\text{SU}(N/2)$ global symmetry, for $N = 2, \dots, 14$. In this plot $c_{\text{free}} = 1/120$, corresponding to the central charge of a free scalar. Here we have taken $k = 11$.

which increases the central charge. In order to demonstrate that the bound does in fact approach the free value, in figure 4.16 we also show the bound for $k = 11$ where $(d - 1)$ has been placed on a logarithmic scale.

We extend these bounds to the situation where Φ^i transforms as a fundamental under an $\text{SU}(N)$ global symmetry in figure 4.17, where we have taken $k = 10$ and show curves for $N = 2, \dots, 14$. All the curves interpolate to the free values Nc_{chiral} as $d \rightarrow 1$, in all cases with a very sharp drop in the bound close to 1. Again we see that the bounds scale linearly with N , and moreover the linear behavior extends out to larger values of d compared to the non-supersymmetric

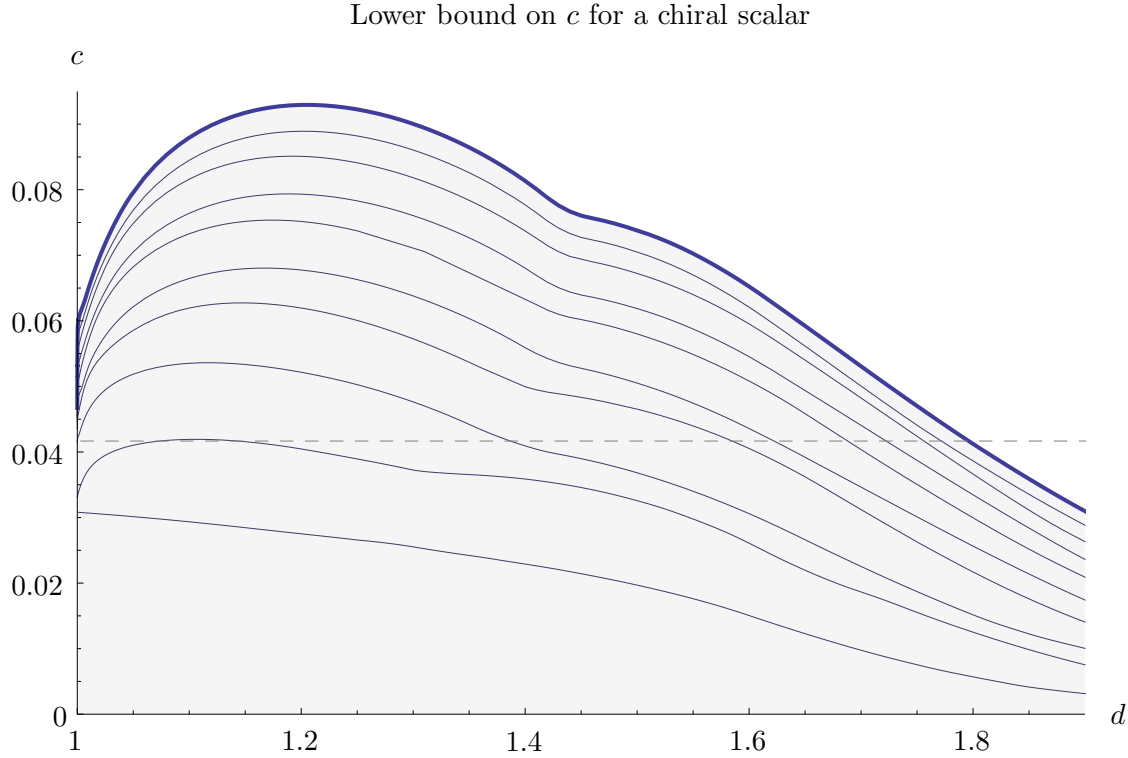


Figure 4.15: A lower bound on the central charge of any SCFT containing a chiral scalar Φ of dimension d . The dashed line is at $c_{\text{chiral}} = 1/24$, corresponding to the central charge of a free chiral superfield ($d = 1$). Despite appearances at this zoom level, all the curves above drop sharply near $d = 1$ and interpolate smoothly to the free value. In this plot we have taken $k = 2, \dots, 11$.

bounds of figure 4.14.

Let us now take a moment to compare these bounds to some concrete SCFTs. The reason that such a comparison is possible is that both d and c are calculable in terms of the $U(1)_R$ symmetry — d is calculable because the dimensions of chiral superconformal primary operators are related to their R charge as $d = \frac{3}{2}R$, and c is calculable via 't Hooft anomaly matching using the relation $c = \frac{1}{32}(9\text{Tr}R^3 - 5\text{Tr}R)$ [112, 113]. The $U(1)_R$ symmetry can then often be determined using symmetry arguments, or more generally using a -maximization [114].

One of the simplest $\mathcal{N} = 1$ SCFTs is supersymmetric QCD with gauge group $SU(N_c)$ and N_f flavors of quarks Q, \bar{Q} in the conformal window $\frac{3}{2}N_c \leq N_f \leq 3N_c$ [103]. In this case the

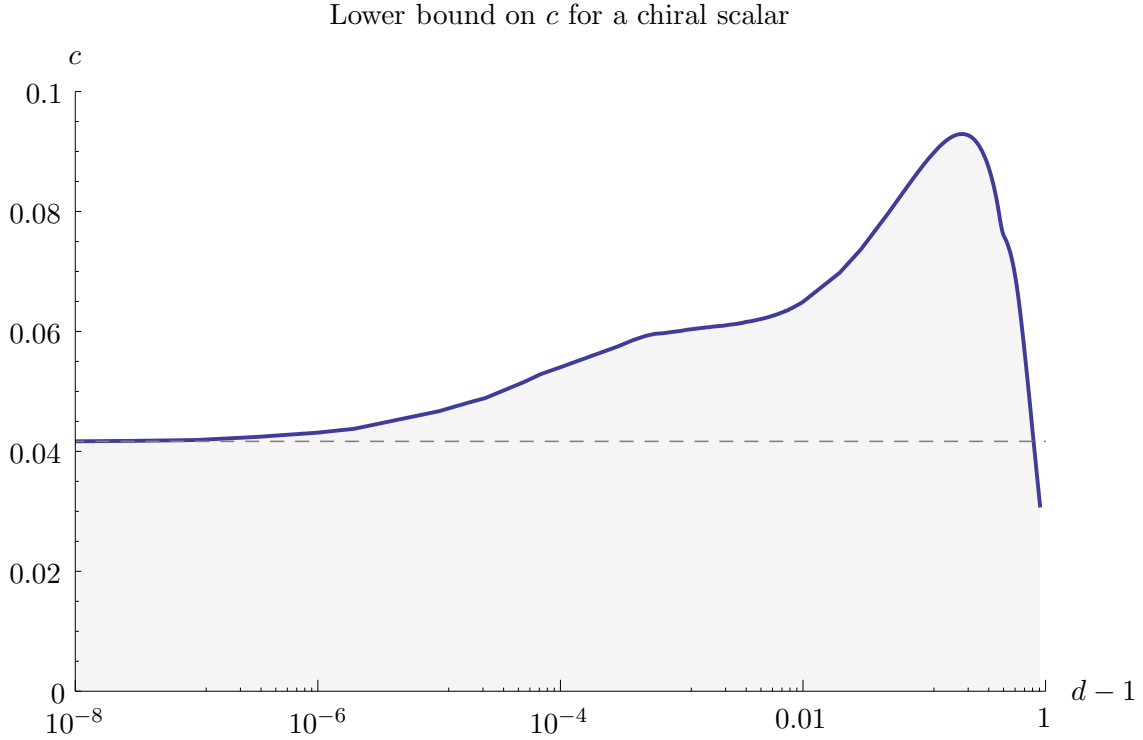


Figure 4.16: The $k = 11$ curve of figure 4.15, where $(d - 1)$ has been placed on a logarithmic scale. The bound smoothly approaches the free value $c_{\text{chiral}} = 1/24$ very close to $d = 1$.

gauge-invariant mesons $M = Q\tilde{Q}$ have $d_M = 3(1 - N_c/N_f)$, while the central charge is evaluated as $c = \frac{1}{16}(7N_c^2 - 9N_c^4/N_f^2 - 2)$. The mesons are bi-fundamentals under the $\text{SU}(N_f) \times \text{SU}(N_f)$ symmetry group, so our bounds will apply by considering either of these groups.

However, we immediately see that the central charge in SQCD grows like $O(N^2)$, so theories at large values of $N_f \sim N_c$ trivially satisfy the bounds. On the other hand, all of the small N theories still have a central charge larger than $1 = 24c_{\text{chiral}}$, so the bound is also easily satisfied for these theories. Part of the problem is that we have only included a subgroup of the full $\text{SU}(N_f) \times \text{SU}(N_f)$ global symmetry when deriving our bounds. In a future publication we hope to extend the bounds to bi-fundamentals transforming under an $\text{SU}(N) \times \text{SU}(N)$ symmetry group, in order to make closer contact with the values realized in SQCD and similar theories.

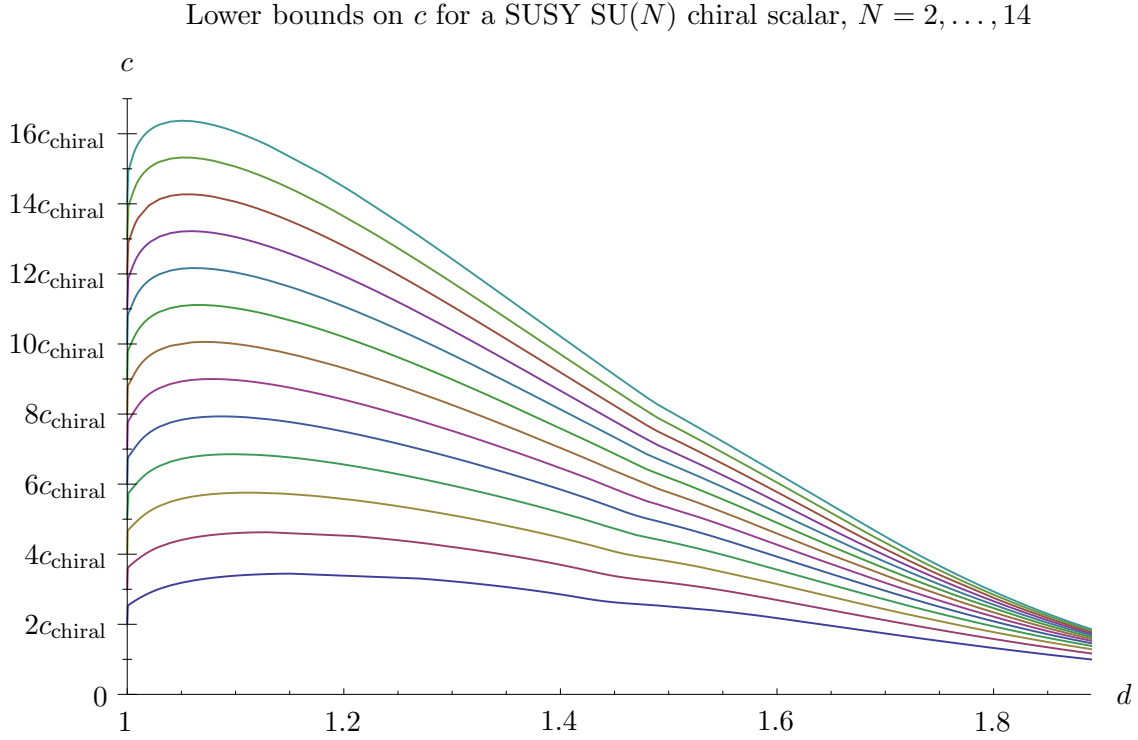


Figure 4.17: Lower bound on the central charge of any SCFT containing a chiral scalar Φ_i of dimension d transforming as a fundamental of an $SU(N)$ global symmetry, for $N = 2, \dots, 14$. Here $c_{\text{chiral}} = 1/24$ denotes the contribution to c from a free chiral superfield. Despite appearances at this zoom level, all the curves drop sharply very close to $d = 1$ and interpolate continuously to the free values. In this plot we have taken $k = 10$.

4.6 Bounds on Current Two-point Functions

4.6.1 General Theories

Now let us turn to placing bounds on another set of fundamental OPE coefficients, namely those appearing in front of spin-1 conserved global symmetry currents. In the OPE between $SO(N)$ or $SU(N)$ fundamentals, we should be careful to distinguish between the $SO(N)$ or $SU(N)$ symmetry currents living in the adjoint representation and singlet currents associated to some other global symmetry that we are not considering explicitly.

Adjoint Currents

Let us begin by focusing on the case of adjoint currents. Consider a CFT with some global symmetry, containing a scalar field ϕ_i transforming in some representation of this symmetry. We will denote by T_{ij}^A the generators in this representation. The associated conserved currents transform as global symmetry adjoints. Ward identities completely fix the three-point functions with one current insertion:

$$\langle \phi_i(x_1) \phi_j(x_2) J_\mu^A(x_3) \rangle = -\frac{i}{2\pi^2} T_{ij}^A \frac{x_{12}^{2-2d}}{x_{13}^2 x_{23}^2} Z_\mu, \quad \text{where} \quad Z_\mu \equiv \frac{x_{13\mu}}{x_{13}^2} - \frac{x_{12\mu}}{x_{12}^2}. \quad (4.85)$$

With the above normalizations, the two-point function $\langle J^A J^B \rangle$ contains undetermined coefficients τ^{AB} that roughly measure the amount of stuff charged under the global symmetry:

$$\langle J^{A\mu}(x_1) J^{B\nu}(x_2) \rangle = \frac{3\tau^{AB}}{4\pi^4} \frac{I^{\mu\nu}(x_{12})}{x_{12}^6}. \quad (4.86)$$

Let us write $\tau^{AB} \equiv \kappa \text{Tr}(T^A T^B)$, where κ can be viewed as a symmetry current ‘central charge.’ As we did for the energy momentum tensor, we can rescale J^A to have a canonically normalized two-point function and absorb κ into the OPE coefficient λ_J^2 associated with the current. In the end, the contribution of an adjoint current to a four-point function of ϕ_i ’s can be written

$$x_{12}^{2d} x_{34}^{2d} \langle \phi_i \phi_j \phi_k \phi_l \rangle \sim \frac{1}{6\kappa} \text{Tr}(T^A T^B)^{-1} T_{ij}^A T_{kl}^B g_{3,1}. \quad (4.87)$$

In order to proceed further we need to specify the global symmetry group. For instance, for $\text{SO}(N)$ and ϕ_i in the vector representation, one can show that

$$\text{Tr}(T^A T^B)^{-1} T_{ij}^A T_{kl}^B = \frac{1}{2} (\delta_{il} \delta_{jk} - \delta_{ik} \delta_{jl}), \quad (4.88)$$

and consequently, comparing to Eq. (4.13), we have $\lambda_J^2 = \frac{1}{12\kappa}$. Similarly, for $\text{SU}(N)$ and ϕ_i in the fundamental representation, we have

$$\text{Tr}(T^A T^B)^{-1} (T^A)_j^i (T^B)_l^k = \delta_l^i \delta_j^k - \frac{1}{N} \delta_j^i \delta_l^k, \quad (4.89)$$

so that $\lambda_J^2 = \frac{1}{6\kappa}$. These relations hold for currents appearing in OPEs in general CFTs; we will discuss the generalization to $\mathcal{N} = 1$ superconformal theories below. However, first we will consider the situation of singlet currents appearing in the OPE, namely currents corresponding to a global symmetry that is different from the $\text{SO}(N)$ or $\text{SU}(N)$ that we are studying.

Singlet Currents

As mentioned above, the $\text{SO}(N)$ or $\text{SU}(N)$ global symmetry current is not the only conserved spin-1 operator of dimension 3 that can contribute to the four-point function; additional currents, possibly transforming in different representations, may also exist. Clearly the presence of an additional conserved current implies the existence of a global symmetry beyond the one exploited to write the crossing symmetry constraints. The OPE coefficient associated to this operator not only contains the two-point function normalization, but also parametrizes our ignorance about the nature of the additional global symmetry. Indeed, when the global symmetry is not specified the three-point function coefficient could in principle be arbitrary.

In the case of fundamentals transforming under an $\text{SO}(N)$ global symmetry, spin-1 operators appearing in the OPE can only transform in the adjoint (antisymmetric) representation, corresponding to the $\text{SO}(N)$ current itself. In the case of $\text{SU}(N)$ fundamentals, along with the adjoint current we also have the possibility of $\text{SU}(N)$ singlet currents.¹⁴ For example, we can think about a CFT with a global symmetry $\text{SU}(N) \times \mathcal{G}$. If we consider scalar operators transforming in some representation of \mathcal{G} with generators \mathcal{T}^A , then the \mathcal{G} -current is a singlet with respect to $\text{SU}(N)$, and its contribution to the four-point function will be

$$\frac{1}{6\kappa_{\mathcal{G}}} \text{Tr}(\mathcal{T}^A \mathcal{T}^B)^{-1} \mathcal{T}_{ij}^A \mathcal{T}_{lm}^B g_{3,1} = \lambda_J^2 \delta_{ij} \delta_{lm} g_{3,1}, \quad (4.90)$$

¹⁴In addition, the OPE $\phi_i \times \phi_j$ could contain conserved spin-1 operators transforming in the antisymmetric representation of $\text{SU}(N)$. However, such currents (along with their complex conjugates) would generate charges which enhance $\text{SU}(N)$ to a larger group $\text{SU}(N) \rightarrow \text{SO}(2N)$. Thus, such theories necessarily fall under the class of CFTs with a global $\text{SO}(2N)$ symmetry, which we consider separately.

where $\kappa_{\mathcal{G}}$ is the two-point function of the \mathcal{G} -current. Until we additionally specify the \mathcal{G} symmetry group and charges, this parameter is arbitrary. However, we can collectively define, by analogy with the adjoint current, an *effective* current two-point function normalization $\kappa_{\text{eff}} \equiv 1/6\lambda_J^2$. We will place bounds on κ_{eff} when we give our results below.

Free Theory and Numerical Results

To clarify the above discussion, let us analyze in detail the theory of N free complex scalars, using only information about the $\text{SU}(N)$ global symmetry, which is contained in the larger $\text{SO}(2N)$ symmetry of the theory. The OPE $\phi_i \times \phi^{\bar{j}\dagger}$ contains an adjoint current and a singlet current, both conserved:

$$J_{\text{Ad}}^A \sim \phi^\dagger T^A \vec{\partial} \phi, \quad J_S \sim \phi^\dagger \vec{\partial} \phi. \quad (4.91)$$

The conformal block decomposition of the scalar four-point function directly gives us the values of the singlet and adjoint OPE coefficients for spin- ℓ currents:

$$\lambda_{\text{Ad}}^2 = \frac{(\ell!)^2}{(2\ell)!}, \quad \kappa = \frac{1}{3}, \quad (4.92)$$

$$\lambda_S^2 = \frac{1}{N} \frac{(\ell!)^2}{(2\ell)!}, \quad \kappa_{\text{eff}} = \frac{N}{3}. \quad (4.93)$$

The first point that we notice is the different scaling of the two above quantities with the size of the symmetry group. While the adjoint current two-point function normalization is independent of N , the singlet one grows with the dimension of the representation. We therefore expect lower bounds on κ_{eff} to scale with N , similarly to the way that the central charge bounds did in the previous section.

Let us now discuss the same theory, using the whole $\text{SO}(N)$ global symmetry. This time only the adjoint current contributes to the four-point function, and its OPE coefficient (along with the other spin- ℓ adjoint operators) can be determined from the conformal block decomposition (see

for instance [59]) as

$$\lambda_{\text{Ad}}^2 = \frac{(\ell!)^2}{(2\ell)!}, \quad \kappa = \frac{1}{6}. \quad (4.94)$$

Now that we have an intuition for the free values of κ and κ_{eff} , we are ready to present numerical bounds in several classes of theories. In figure 4.18, we show a lower bound on the two-point function coefficient κ for a CFT with an $\text{SO}(N)$ global symmetry for $N = 2, \dots, 14$. As expected, when $d \rightarrow 1$, all of the bounds drop sharply to the free $\text{SO}(N)$ value $\kappa = 1/6$. The bounds get stronger as N increases, while as d varies away from 1, they first become stronger and then weaken.

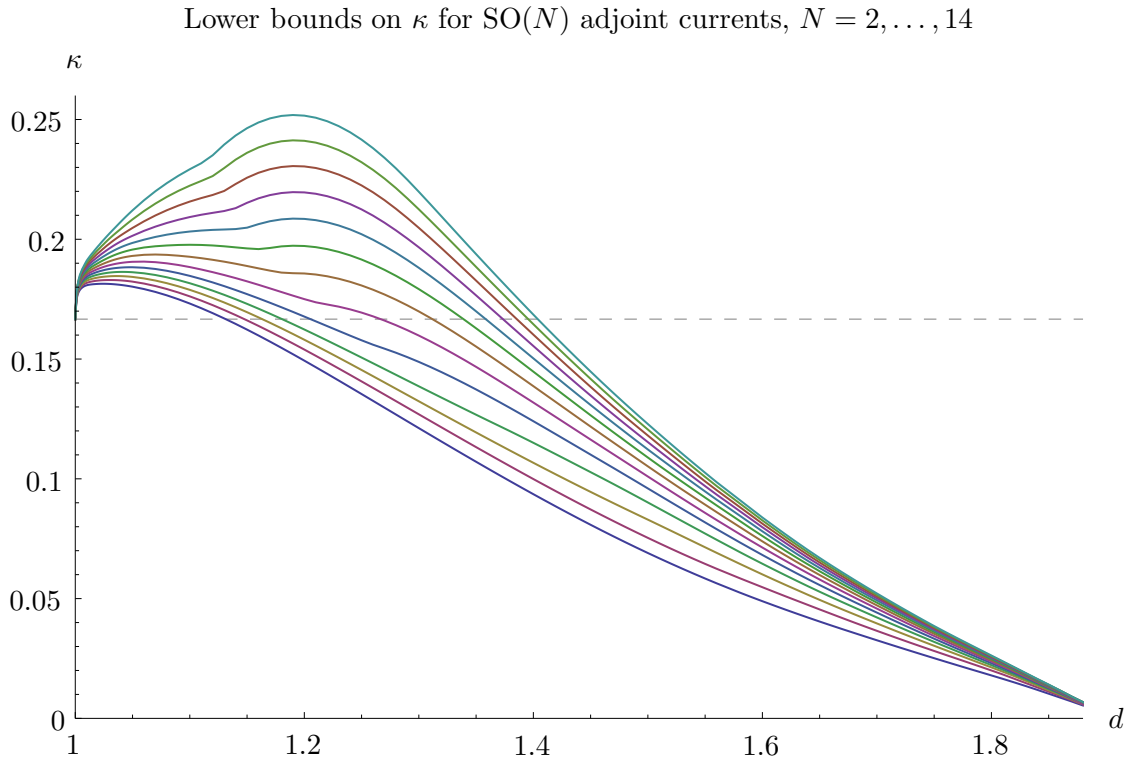


Figure 4.18: A lower bound on the two-point function coefficient $\langle J_\mu^A J_\nu^B \rangle \propto \kappa \text{Tr}(T^A T^B)$ of the $\text{SO}(N)$ adjoint current appearing in $\phi \times \phi$, where ϕ transforms in the fundamental of an $\text{SO}(N)$ global symmetry group. All curves smoothly approach the free $\text{SO}(N)$ value $\kappa = 1/6$. Here we have taken $k = 11$.

As a second example, in figure 4.19 we consider the case of an $\text{SU}(N)$ global symmetry

and present lower bounds on κ_{eff} for a singlet current. Our expectation that the constraints scale almost linearly with N (when d is close to 1) is confirmed. Thus, this quantity serves as a rough measure of the number of degrees of freedom in the theory transforming under the symmetry, at least near $d = 1$. On the other hand, the linear scaling disappears as d increases.

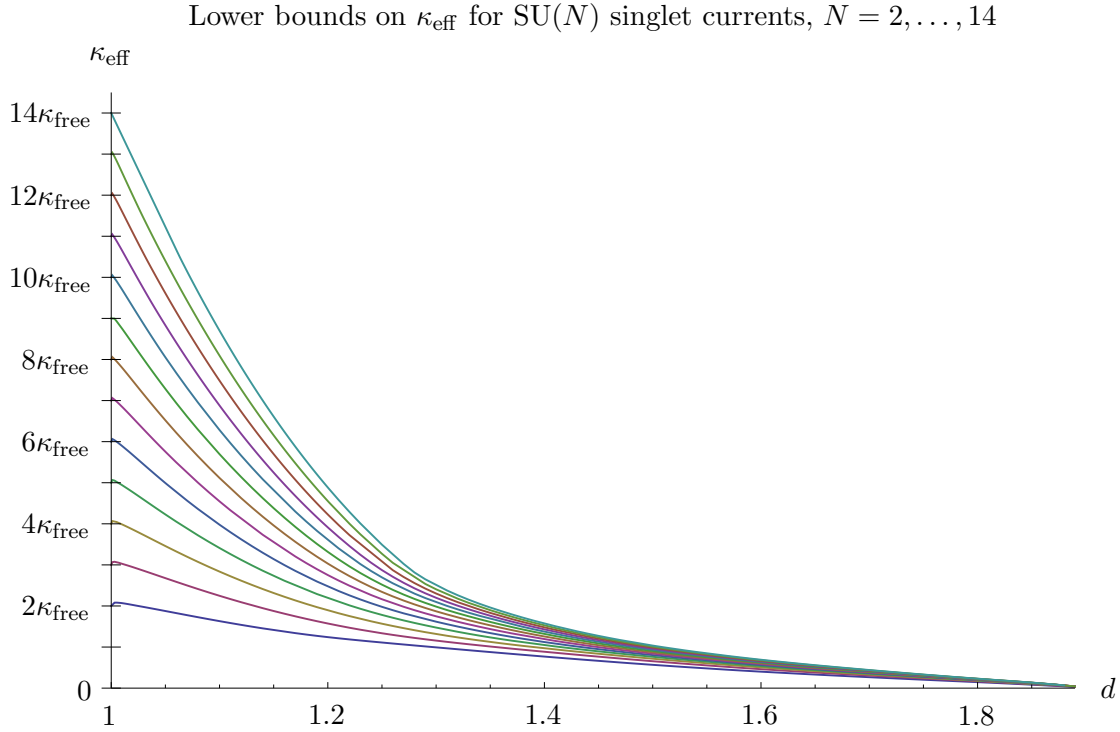


Figure 4.19: A lower bound on the effective two-point function coefficient $\kappa_{\text{eff}} = 1/6\lambda_J^2$ of $\text{SU}(N)$ singlet currents appearing in $\phi_i \times \phi^{\bar{j}\dagger}$, where ϕ_i transforms in the fundamental of an $\text{SU}(N)$ global symmetry group. All curves interpolate continuously to the free values $N\kappa_{\text{free}}$ where $\kappa_{\text{free}} = 1/3$, and in this plot we have taken $k = 11$.

4.6.2 Superconformal Theories

Let us generalize the above bounds to theories with $\mathcal{N} = 1$ supersymmetry, where currents are descendants of scalar superconformal primaries of dimension 2. Consider four-point functions $\langle \Phi_i \Phi^{\bar{j}\dagger} \Phi_k \Phi^{\bar{l}\dagger} \rangle$ of chiral and anti-chiral operators transforming under an $\text{SU}(N)$ global symmetry.

SU(N) adjoint currents give a superconformal block contribution

$$x_{12}^{2d} x_{34}^{2d} \langle \Phi_i \Phi^{\bar{j}\dagger} \Phi_k \Phi^{\bar{l}\dagger} \rangle \sim \frac{1}{\kappa} \text{Tr}(T^A T^B)^{-1} (T^A)_i^{\bar{j}} (T^B)_k^{\bar{l}} \mathcal{G}_{2,0}, \quad (4.95)$$

while SU(N) singlet currents give an effective superconformal block contribution

$$x_{12}^{2d} x_{34}^{2d} \langle \Phi_i \Phi^{\bar{j}\dagger} \Phi_k \Phi^{\bar{l}\dagger} \rangle \sim \frac{1}{\kappa_{\text{eff}}} \delta_i^{\bar{j}} \delta_k^{\bar{l}} \mathcal{G}_{2,0}. \quad (4.96)$$

In figure 4.20, we show bounds on κ for adjoint currents appearing in $\Phi_i \times \Phi^{\bar{j}\dagger}$, for SCFTs with an SU(N) global symmetry and $N = 2, \dots, 14$. These bounds again increase strongly with N , growing as a roughly affine function. For $d \lesssim 1.5$, κ must be substantially higher than its free value, with the bound dropping sharply to the contribution of a free chiral superfield $\kappa_{\text{chiral}} = 1$ near $d = 1$. Consequently, the free theory appears to be isolated in the space of SCFTs with an SU(N) flavor symmetry. This accords with our intuition from theories with a Lagrangian description. To couple a free SU(N) fundamental to a nontrivial interacting sector (and thus raise its dimension away from $d = 1$), we need additional matter which must itself transform under SU(N).

In figure 4.21, we also show a lower bound on κ_{eff} for singlet currents appearing in $\Phi_i \times \Phi^{\bar{j}\dagger}$. Once again, we see that these bounds increase with N , scaling roughly linearly for small d . As in the adjoint case above, the bounds drop very sharply to their free values $N\kappa_{\text{chiral}}$ near $d = 1$, while the N scaling disappears as d increases.

Comparison to SQCD

As with central charges, our bounds on current two-point functions can be checked explicitly in a given superconformal theory. For example, in SUSY QCD, SU(N_f) $_L$ and SU(N_f) $_R$ flavor currents appear in the OPE of a chiral meson and its conjugate

$$M_i^{\dagger\bar{i}} \times M_j^{\bar{j}} \sim \delta_j^i (T^A)_i^{\bar{j}} J_R^A + \delta_i^{\bar{j}} (T^A)_j^i J_L^A + \dots \quad (4.97)$$

Here, i, j are indices for SU(N_f) $_L$ and \bar{i}, \bar{j} are indices for SU(N_f) $_R$. We have not yet generated bounds that exploit the full SU(N_f) $_L \times$ SU(N_f) $_R$ symmetry group of SQCD. However, we can

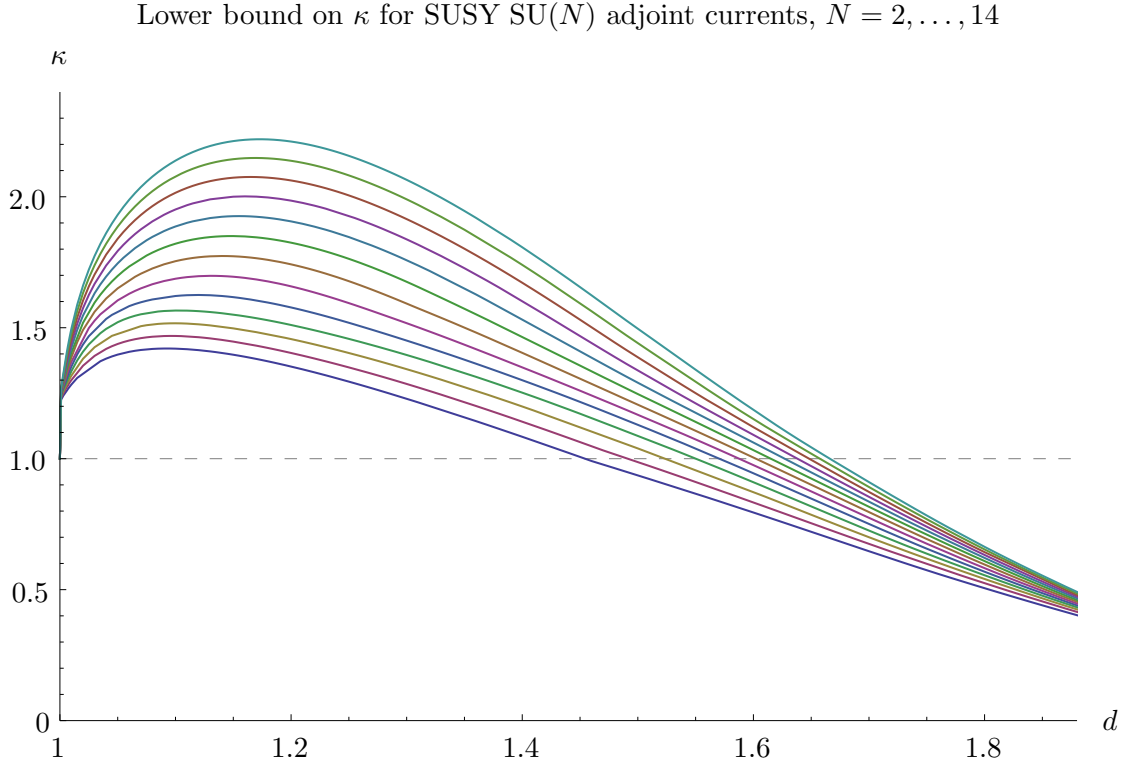


Figure 4.20: A lower bound on the two-point function coefficient $\langle J^A J^B \rangle \propto \kappa \text{Tr}(T^A T^B)$ of an $SU(N)$ adjoint current appearing in $\Phi_i \times \Phi^{\dagger j}$, where Φ_i is a chiral scalar transforming in the fundamental of an $SU(N)$ global symmetry group in an SCFT. Despite appearances at this zoom level, all the curves above drop sharply near $d = 1$ and interpolate continuously to the free value $\kappa_{\text{chiral}} = 1$. Here we have taken $k = 10$.

compare to our $SU(N)$ bounds by ‘forgetting’ one of the flavor groups, say $SU(N_f)_R$, and examining the theory from the point of view of $SU(N_f)_L$ alone. Specifically, we shall set $\tilde{\nu} = \tilde{j} = 1$, so that the right-flavor currents J_R^A are then singlet scalars in $M_1^{\dagger i} \times M_j^1$, while the left-flavor currents J_L^A are adjoints.

The current two-point functions for J_R^A and J_L^A in SQCD both scale like N_f (or N_c). However, only our $SU(N)$ -singlet bounds scale with N , and thus have a chance of approaching the values for SQCD. Consequently, we will focus on the contribution of J_R^A to the conformal block

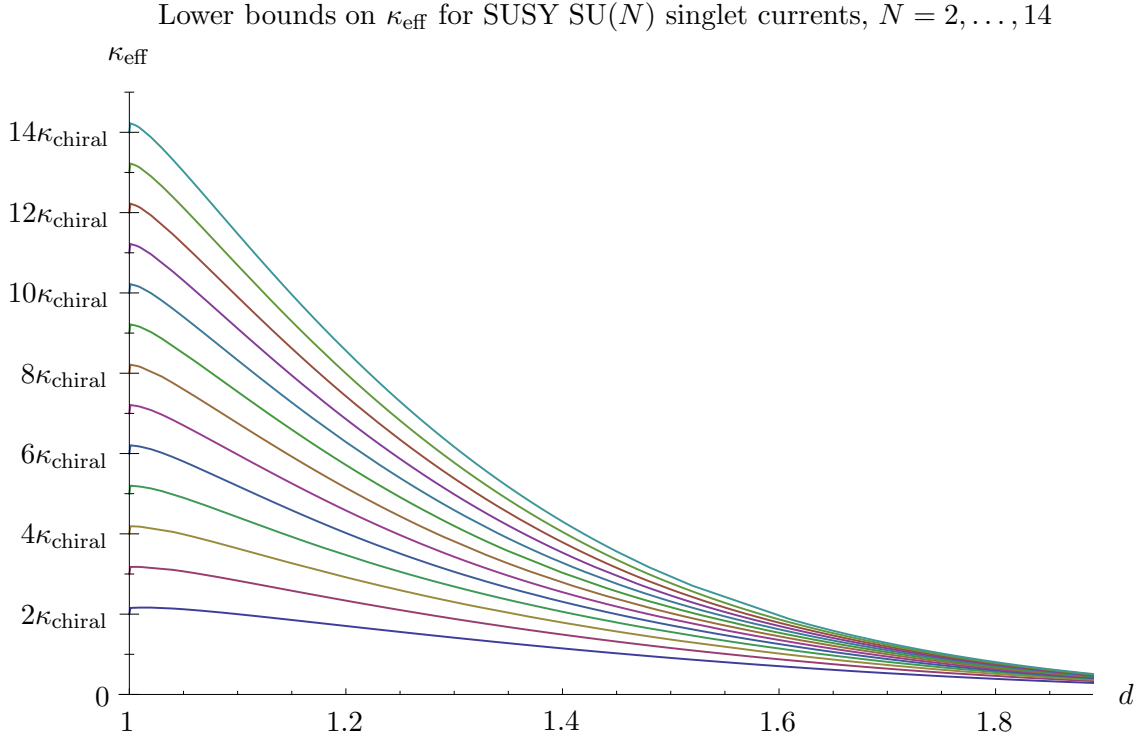


Figure 4.21: A lower bound on the effective two-point function coefficient $\kappa_{\text{eff}} = 1/\lambda_J^2$ of $\text{SU}(N)$ singlet currents appearing in $\Phi_i \times \Phi_j^\dagger$, where Φ_i is a chiral scalar transforming in the fundamental of an $\text{SU}(N)$ global symmetry group. Despite appearances at this zoom level, all the curves above drop sharply near $d = 1$ and interpolate continuously to the free values $N\kappa_{\text{chiral}}$ where $\kappa_{\text{chiral}} = 1$. Here we have taken $k = 10$.

expansion of meson four-point functions. This reads

$$x_{12}^{2d} x_{34}^{2d} \langle M_1^{\dagger i} M_j^1 M_l^{\dagger k} M_l^1 \rangle = \tau_{AB} (T^A)_1^1 (T^B)_1^1 \delta_j^i \delta_l^k \mathcal{G}_{2,0} + \dots, \quad (4.98)$$

where $\mathcal{G}_{2,0}$ is the superconformal block for a conserved current multiplet and $\tau_{AB} = (\tau^{AB})^{-1}$ is the inverse two-point function coefficient for J_R^A . In superconformal theories, τ^{AB} can be computed simply in terms of 't Hooft anomalies using $\tau^{AB} = -3\text{Tr}(RT^A T^B)$. For J_R^A , this becomes

$$\langle J_R^A J_R^B \rangle \propto \tau^{AB} = \frac{3N_c^2}{2N_f} \delta^{AB}, \quad (4.99)$$

where the $SU(N_f)$ generators are normalized according to $\text{Tr}(T^A T^B) = \frac{1}{2} \delta^{AB}$. Thus, we have

$$\frac{1}{\kappa_{\text{eff}}} = \frac{N_f}{3N_c^2} \left(\delta_1^1 \delta_1^1 - \frac{1}{N_f} \delta_1^1 \delta_1^1 \right) = \frac{N_f - 1}{3N_c^2}. \quad (4.100)$$

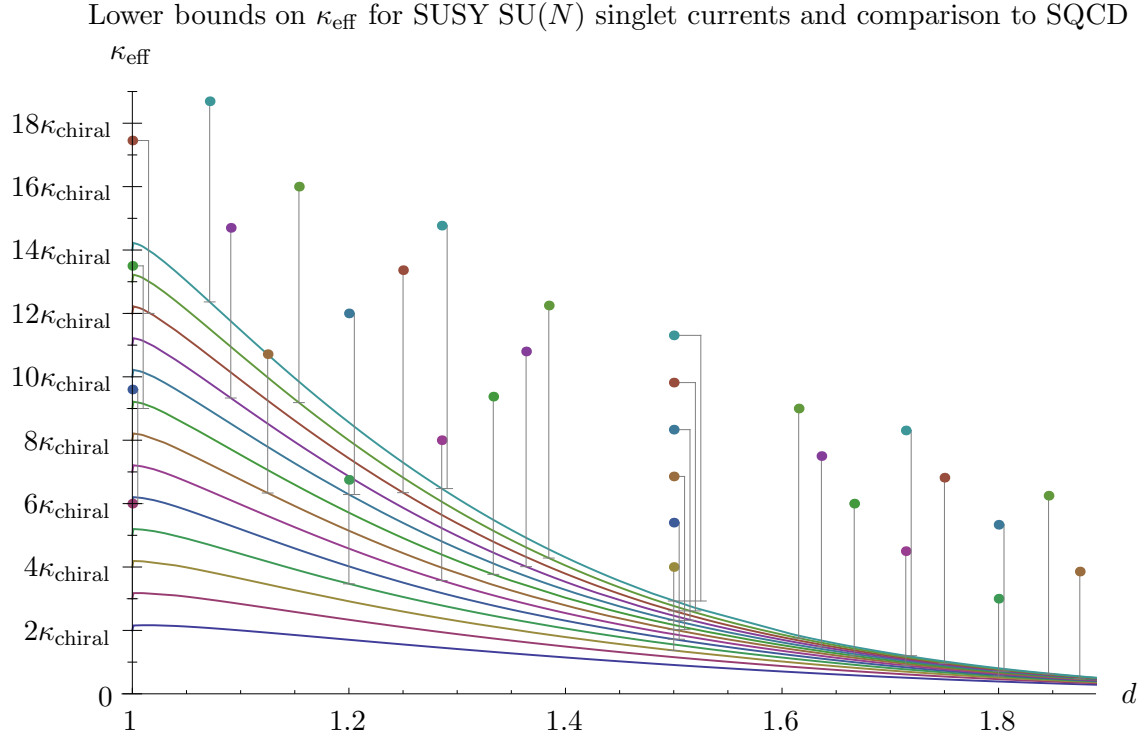


Figure 4.22: A lower bound on the effective two-point function coefficient $\kappa_{\text{eff}} = 1/\lambda_J^2$ of $SU(N)$ singlet currents appearing in $\Phi_i \times \Phi^{\dagger j}$, where Φ_i is a chiral scalar transforming in the fundamental of an $SU(N)$ global symmetry group. Here we have taken $k = 10$. We have also plotted points corresponding to SQCD theories with various values of N_f and N_c . The lines below each point indicate the distance to the corresponding bound. Many SQCD theories lie within an $O(1)$ factor from our bounds.

In figure 4.22 we compare this value of κ_{eff} for several SQCD theories to our singlet current bounds from figure 4.21. For many values of N_f and N_c , our bound comes within an $O(1)$ factor of the SQCD value, with the smallest separation at small dimensions $d \sim 1$. We expect our bound to become stronger with the added information of $SU(N_f)_R$ symmetry, perhaps resulting in a hybrid of figures 4.21 and 4.20. It will be interesting to compare SQCD to these new bounds, and understand more about the structure of four-point functions in this important theory.

4.7 Conclusions

Let us briefly summarize our main results. In this work we explored bounds on operator dimensions and OPE coefficients in 4D CFTs and $\mathcal{N} = 1$ SCFTs, building on the previous studies performed in [54–60]. These bounds can be viewed as the initial stages of a concrete implementation of a 4D conformal bootstrap program. Here we focused on bounds in the presence of $\text{SO}(N)$ and $\text{SU}(N/2)$ global symmetries, which had previously shown themselves to be more difficult (but not impossible [60]) to obtain using algorithms based on linear programming methods. In order to push the program further, we presented a new algorithm based on semidefinite programming, which utilized the fact that derivatives of conformal blocks can be arbitrarily well approximated by positive functions times polynomials in the operator dimensions. This new algorithm enabled us to show that there are completely general bounds on CFTs and SCFTs in the presence of global symmetries that are significantly stronger than were previously known to exist.

In particular, we greatly strengthened bounds on dimensions of singlet operators appearing in the OPE between fundamentals transforming under $\text{SO}(N)$ or $\text{SU}(N/2)$ global symmetries. Bounds on dimensions of singlet operators in the presence of $\text{SO}(4)$ or $\text{SU}(2)$ global symmetries are relevant for models of conformal technicolor, and our bounds place severe constraints on these models, particularly when one does not assume any special flavor structure in four-fermion operators. We refer readers to [109] for further discussion of these constraints. In fact, in the present work we saw that bounds on singlet operators were in general identical between $\text{SO}(N)$ and $\text{SU}(N/2)$ global symmetries. We have so far not been able to construct a rigorous proof of this equivalence, so it would be good to gain a better understanding of it in future work.

We also obtained similar bounds on operator dimensions in $\mathcal{N} = 1$ SCFTs, where we showed that there are bounds on the lowest-dimension scalar appearing in the $\Phi \times \Phi^\dagger$ OPE that appear to asymptote to the line $\Delta = 2d$ near $d \sim 1$. This result is particularly interesting in light of

the discussion of [102] on positive anomalous dimensions of these operator in SCFTs — our results demonstrate that this should not be possible when one is sufficiently close to the free limit.

In this work we also initiated an exploration of both upper and lower bounds on OPE coefficients of protected operators appearing in the $\Phi \times \Phi$ OPE in SCFTs. In this case, lower bounds are possible due to the fact that there is a gap in the dimensions of operators appearing in this OPE that is required by unitarity. Because one can obtain bounds in both directions, we are able to see that the possible behavior is very tightly constrained even when one is only somewhat close to the free limit. We expect that similar lower bounds should be possible in any situation (including non-supersymmetric theories) where one assumes that there is a dimension gap such that only a single operator can contribute to the conformal block decomposition up to a certain dimension.

We also explored bounds on central charges and current two-point function coefficients in the presence of operators transforming as fundamentals under $\text{SO}(N)$ or $\text{SU}(N/2)$ global symmetries, finding bounds that scale linearly with N when the operator dimension is close to 1. An exception is the case when the current is the adjoint current corresponding to the $\text{SO}(N)$ or $\text{SU}(N/2)$ symmetry itself, in which case the bounds approach a value independent of N in the free limit. In superconformal theories, these bounds can be compared to concrete theories where the central charge c and current two-point functions κ are calculable using 't Hooft anomaly matching. While the central charge bounds are still relatively far from their realized values, we showed concretely that our bounds on κ are an $O(1)$ amount away from the values realized in supersymmetric QCD in the conformal window.

A clear future direction is to generalize these bounds on $\mathcal{N} = 1$ SCFTs to situations with bi-fundamentals transforming under $\text{SU}(N) \times \text{SU}(N)$ global symmetries (or adjoints transforming under $\text{SU}(N)$ global symmetries). Then one would hope to see bounds on the central charge that scale like $\sim N^2$, as well as significantly stronger bounds on current two-point functions. It will be

fascinating to see how these bounds compare to concrete $\mathcal{N} = 1$ theories such as supersymmetric QCD in the conformal window, particularly if one can find theories that nearly saturate the bounds. One could also input all known information about these theories and attempt to find even stronger constraints on the dimensions of unprotected operators.¹⁵ We plan to explore these bounds in a future publication.

Another interesting direction would be to apply these methods to four-point functions of operators with spin, such as symmetry currents or the stress tensor. With the methods of chapter 2, this is now a possibility.

Of course, it would be nice to have a better analytical understanding of the structure of the optimal bounds. While such an understanding has eluded us so far, it is possible that a new approach (such as studying the Mellin representation as in [81, 82, 85]) could shed light on the origin of these bounds. Less ambitiously, it would be good to study whether expansions of the crossing relation around other points in (z, \bar{z}) space may provide a more efficient way to find an optimal linear functional. A related question is to understand whether any of the multiple crossing relations that we have used in cases of global symmetries are redundant or unnecessary for obtaining an optimal bound. We leave such questions to future work.

Finally, we hope that progress can be made at understanding where these bounds fit in the context of the AdS/CFT correspondence [41–43]. Bounds on the central charge and current two-point function coefficients can be mapped to limitations on the strength of gravitational or gauge forces in the presence of light bulk excitations. In the present work, we have obtained bounds that scale with the sizes of global symmetry representations, which in AdS corresponds to scaling with the size of the bulk gauge group. While many of our bounds necessarily apply in a highly quantum regime, we have seen that there are at least some bounds (e.g., bounds on operator dimensions in

¹⁵An alternate approach to learning about these dimensions is to look for hidden structure such as integrability (e.g., see [115]) that makes the theory more solvable than one naïvely expects. We recently started exploring the possibility of such structure in $\mathcal{N} = 1$ SQCD in [116].

SCFTs) that constrain deviations from the large- N factorization limit, where an AdS description would be weakly coupled. It would then be good to find alternate ways of arriving at these bounds in the context of AdS, particularly since these constraints are not obvious from the perspective of effective field theory [102]. One hopes that thinking more along these lines will lead to a deeper understanding of which low-energy theories may admit consistent UV completions, particularly in the context of quantum gravity.

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4.A Polynomial Approximation Details

In this appendix we give further details of our implementation of the optimization problem discussed in section 4.2.5 using semidefinite programming. In all of the situations we consider, the problem is to find the optimal set of coefficients a_{mnk} , which minimizes the combination $-a_{mnk}V_0^{S^+,mnk}(0)$, subject to the constraints

$$a_{mnk}V_{\ell_0}^{I_0,mnk}(\Delta_0) = 1, \tag{4.101}$$

$$a_{mnk}V_{\ell}^{I,mnk}(\Delta) \geq 0, \quad \text{for all other (non-unit) operators in the spectrum.} \tag{4.102}$$

Here $V_\ell^{I,mnk}(\Delta) = \partial_z^m \partial_{\bar{z}}^n V_{\Delta,\ell}^{I,k}$ denotes derivatives of the k -th component of the appropriate vector $V_{\Delta,\ell}^I$, which may be any of the functions $\{F_{\Delta,\ell}, H_{\Delta,\ell}, \mathcal{F}_{\Delta,\ell}, \mathcal{H}_{\Delta,\ell}, \tilde{\mathcal{F}}_{\Delta,\ell}, \tilde{\mathcal{H}}_{\Delta,\ell}\}$. The index I denotes possible global symmetry representations.

As discussed in section 4.2.5, to apply semidefinite programming we must approximate $V_\ell^{I,mnk}(\Delta)$ as $\chi_\ell(\Delta) P_\ell^{I,mnk}(\Delta)$, where $\chi_\ell(\Delta)$ is a strictly positive function, and $P_\ell^{I,mnk}(\Delta)$ is a polynomial in Δ . Let us begin by discussing derivatives of $F_{\Delta,\ell}$ and $H_{\Delta,\ell}$. It is convenient to first rescale each of these by a (Δ, ℓ) -independent function of z and \bar{z} , so that they become sums of terms that factorize:

$$\begin{aligned} E_{\Delta,\ell,+}(z, \bar{z}) &\equiv \left[\frac{(z - \bar{z})}{[(1-z)(1-\bar{z})]^d} - \frac{(z - \bar{z})}{(z\bar{z})^d} \right] F_{\Delta,\ell}(z, \bar{z}) \\ &= \left[\frac{k_{\Delta+l}(z)k_{\Delta-l-2}(\bar{z})}{(z\bar{z})^{d-1}} + \frac{k_{\Delta+l}(1-z)k_{\Delta-l-2}(1-\bar{z})}{[(1-z)(1-\bar{z})]^{d-1}} \right] - (z \leftrightarrow \bar{z}), \end{aligned} \quad (4.103)$$

$$\begin{aligned} E_{\Delta,\ell,-}(z, \bar{z}) &\equiv \left[\frac{(z - \bar{z})}{[(1-z)(1-\bar{z})]^d} + \frac{(z - \bar{z})}{(z\bar{z})^d} \right] H_{\Delta,\ell}(z, \bar{z}) \\ &= \left[\frac{k_{\Delta+l}(z)k_{\Delta-l-2}(\bar{z})}{(z\bar{z})^{d-1}} - \frac{k_{\Delta+l}(1-z)k_{\Delta-l-2}(1-\bar{z})}{[(1-z)(1-\bar{z})]^{d-1}} \right] - (z \leftrightarrow \bar{z}), \end{aligned} \quad (4.104)$$

where $k_\beta(z) \equiv z^{\beta/2} {}_2F_1(\beta/2, \beta/2, \beta, z)$. Derivatives of these quantities at $(1/2, 1/2)$ can then be straightforwardly evaluated using [57]

$$\begin{aligned} C_{\beta,d}^n &\equiv \partial_z^n \left[z^{1-d+\beta/2} {}_2F_1(\beta/2, \beta/2, \beta, z) \right]_{z=1/2} \\ &= 2^{n+(d-1)-\beta/2} \frac{\Gamma(\beta/2 + 2 - d)}{\Gamma(\beta/2 + 2 - d - n)} {}_3F_2(\beta/2 + 2 - d, \beta/2, \beta/2; \beta/2 + 2 - d - n, \beta; 1/2) \\ &= 2(5 - 2d - n)C_{\beta,d}^{n-1} + 2(\beta(\beta - 2) + 2n(n - 3) - 2d^2 + 8d - 2)C_{\beta,d}^{n-2} \\ &\quad + 8(n - 2)(n + d - 4)^2 C_{\beta,d}^{n-3} \\ &= P_d^n(\beta)k_\beta(1/2) + Q_d^n(\beta)k'_\beta(1/2). \end{aligned} \quad (4.105)$$

Here $P_d^n(\beta)$ and $Q_d^n(\beta)$ are polynomials in β that can be determined through the above recursion relation for $C_{\beta,d}^n$.¹⁶ Note that taking $z \rightarrow 1 - z$ simply introduces an overall factor of $(-1)^n$.

¹⁶This recursion relation follows from the hypergeometric differential equation for $k_\beta(z)$, which itself is a consequence of the fact that conformal blocks are eigenfunctions of the quadratic Casimir of the conformal group.

In Eq. (4.105), we have written derivatives of $k_\beta(z)$ at $z = 1/2$ in terms of polynomials in β , up to two non-polynomial quantities: $\beta k_\beta(1/2)$ and $k'_\beta(1/2)$. For the purposes of writing positivity constraints, we are free to divide by $k'_\beta(1/2)/\beta$, which is positive for all β that occur in unitary theories ($\beta \geq -1$). Now, the crucial fact for us is that the remaining non-polynomial quantity $K_\beta \equiv \beta k_\beta(1/2)/k'_\beta(1/2)$ is meromorphic in β , and admits a simple approximation in terms of rational functions

$$K_\beta \equiv \frac{\beta k_\beta(1/2)}{k'_\beta(1/2)} \simeq \frac{1}{\sqrt{2}} \prod_{j=0}^M \frac{(\beta - r_j)}{(\beta - s_j)} \equiv \frac{N_M(\beta)}{D_M(\beta)}, \quad (4.106)$$

where $N_M(\beta)$ and $D_M(\beta)$ are polynomials in β of degree $M+1$. Here, r_j is the j 'th zero of $\beta k_\beta(1/2)$ and s_j is the j 'th zero of $k'_\beta(1/2)$, both of which are close to $-2j-1$. Ordinarily we would need to account for both the zeros and poles of $\beta k_\beta(1/2)$ and $k'_\beta(1/2)$ in the above product representation. However, the poles of $\beta k_\beta(1/2)$ and $k'_\beta(1/2)$ coincide at the negative odd integers, and so cancel between numerator and denominator.¹⁷

The approximation Eq. (4.106) becomes arbitrarily good as more zeros are included, and moreover converges very quickly. In fact, one can show that

$$r_j, s_j = -1 - 2j + O(2^{-5.5j}) \quad j = 0, 1, 2, \dots, \quad (4.107)$$

so that

$$K_\beta = \frac{N_M(\beta)}{D_M(\beta)} \times \left(1 + O\left(\frac{2^{-5.5M}}{\beta + 2M + 1} \right) \right) \quad (\beta \geq -2M - 1). \quad (4.108)$$

Consequently, it is sufficient to take $M \sim$ a few to achieve an accurate rational approximation for K_β that holds uniformly for all physical values $\beta \geq -1$. In practice, we found that $M = 3$ or 4 gives excellent results, which remain effectively unchanged when M is increased. Henceforth,

¹⁷The factor $1/\sqrt{2}$ is $\lim_{\beta \rightarrow \infty} K_\beta$ (with an arbitrary phase for β), as can be verified using the standard integral formula for ${}_2F_1$ hypergeometric functions. Since this limit exists, K_β is meromorphic on the Riemann sphere, not just \mathbb{C} .

we will assume that some appropriate M has been chosen, and write simply $N(\beta)$ and $D(\beta)$ for brevity.

Combining Eqs. (4.105) and (4.106), we can now write

$$C_{\beta,d}^n = \frac{k'_\beta(1/2)}{\beta D(\beta)} u_d^n(\beta), \quad (4.109)$$

where

$$u_d^n(\beta) \equiv N(\beta) P_d^n(\beta) + \beta D(\beta) Q_d^n(\beta) \quad (4.110)$$

is a polynomial in β , and it can be verified that the pre-factor $k'_\beta(1/2)/\beta D(\beta)$ is positive for all $\beta \geq -1$. Note that the degree of $u_d^n(\beta)$ depends on the number of roots $M+1$ included in the approximation of Eq. (4.106).

Derivatives of $E_{\Delta,\ell,\pm}(z, \bar{z})$ at $(1/2, 1/2)$ can now be written

$$\partial_z^m \partial_{\bar{z}}^n E_{\Delta,\ell,\pm}(1/2, 1/2) = \chi_\ell(\Delta) U_{\ell,d,\pm}^{mn}(\Delta), \quad (4.111)$$

where

$$\chi_\ell(\Delta) \equiv \frac{k'_{\Delta+\ell}(1/2) k'_{\Delta-\ell-2}(1/2)}{(\Delta+\ell)(\Delta-\ell-2) D(\Delta+\ell) D(\Delta-\ell-2)} \quad (4.112)$$

is positive, and

$$U_{\ell,d,\pm}^{mn}(\Delta) \equiv (1 \pm (-1)^{m+n}) [u_d^m(\Delta+\ell) u_d^n(\Delta-\ell-2) - (m \leftrightarrow n)] \quad (4.113)$$

is a polynomial in Δ . The inequalities $a_{mnk} V_\ell^{I,mnk}(\Delta) \geq 0$ given in (4.102) are then equivalent to a set of polynomial inequalities, which can be rewritten in terms of a semidefinite program as described in section 4.2.5.

Next let us consider derivatives of the functions $\mathcal{F}_{\Delta,\ell}(z, \bar{z})$ and $\mathcal{H}_{\Delta,\ell}(z, \bar{z})$, appearing in superconformal crossing relations. We can again take derivatives using Eq. (4.109) after rescaling by

the same functions of z and \bar{z} appearing in Eqs. (4.103) and (4.104). Applying $\partial_z^m \partial_{\bar{z}}^n$ at $(1/2, 1/2)$ to the resulting functions gives

$$\begin{aligned} \chi_\ell(\Delta) \Bigg[& U_{\ell,d,\pm}^{mn}(\Delta) \\ & + \frac{(\Delta + \ell)}{4(\Delta + \ell + 1)} \frac{D(\Delta + \ell)}{D(\Delta + \ell + 2)} \mathcal{K}_{\Delta+\ell} U_{\ell+1,d,\pm}^{mn}(\Delta + 1) \\ & + \frac{(\Delta - \ell - 2)}{4(\Delta - \ell - 1)} \frac{D(\Delta - \ell - 2)}{D(\Delta - \ell)} \mathcal{K}_{\Delta-\ell-2} U_{\ell-1,d,\pm}^{mn}(\Delta + 1) \\ & + \frac{(\Delta + \ell)(\Delta - \ell - 2)}{16(\Delta + \ell + 1)(\Delta - \ell - 1)} \frac{D(\Delta + \ell)D(\Delta - \ell - 2)}{D(\Delta + \ell + 2)D(\Delta - \ell)} \mathcal{K}_{\Delta+\ell} \mathcal{K}_{\Delta-\ell-2} U_{\ell,d,\pm}^{mn}(\Delta + 2) \Bigg], \end{aligned} \quad (4.114)$$

where

$$\mathcal{K}_\beta \equiv \frac{\beta}{\beta + 2} \frac{k'_{\beta+2}(1/2)}{k'_\beta(1/2)}. \quad (4.115)$$

We can then use the fact that \mathcal{K}_β can be arbitrarily well approximated by a rational function

$$\mathcal{K}_\beta \simeq (12 - 8\sqrt{2}) \frac{(\beta + 1) \prod_i (\beta + 2 - s_i)}{\prod_j (\beta - s_j)}. \quad (4.116)$$

Again, the approximation improves as more roots are included, and converges after only a few terms. Thus, by isolating the polynomial numerator and denominator of the quantity

$$\frac{\beta}{4(\beta + 1)} \frac{D(\beta)}{D(\beta + 2)} \mathcal{K}_\beta \equiv \frac{\mathcal{N}(\beta)}{\mathcal{D}(\beta)}, \quad (4.117)$$

we can write the derivatives as a positive function times a polynomial in Δ :

$$\begin{aligned} \frac{\chi_\ell(\Delta)}{\mathcal{D}(\Delta + \ell)\mathcal{D}(\Delta - \ell - 2)} \times \Big[& \mathcal{D}(\Delta + \ell)\mathcal{D}(\Delta - \ell - 2) U_{\ell,d,\pm}^{mn}(\Delta) \\ & + \mathcal{N}(\Delta + \ell)\mathcal{D}(\Delta - \ell - 2) U_{\ell+1,d,\pm}^{mn}(\Delta + 1) \\ & + \mathcal{D}(\Delta + \ell)\mathcal{N}(\Delta - \ell - 2) U_{\ell-1,d,\pm}^{mn}(\Delta + 1) \\ & + \mathcal{N}(\Delta + \ell)\mathcal{N}(\Delta - \ell - 2) U_{\ell,d,\pm}^{mn}(\Delta + 2) \Big]. \end{aligned} \quad (4.118)$$

Finally, let us note that the results for $\tilde{\mathcal{F}}_{\Delta,\ell}(z, \bar{z})$ and $\tilde{\mathcal{H}}_{\Delta,\ell}(z, \bar{z})$ are identical, but with odd-spin terms having the opposite sign. Thus, we see that we can reformulate any of the sum rules appearing in SCFTs as a semidefinite program, following the logic described in section 4.2.5.

4.B Implementation in SDPA-GMP

In this appendix we'll give further details of our implementation of the SDP. As we described in section 4.2.5 and appendix 4.A, the general problem (phrased as a SDP) is to minimize $-a_i V_0^{S^+,i}(0)$, subject to the constraints

$$\begin{aligned} a_i V_{\ell_0}^{I_0,i}(\Delta_0) &= 1, \\ a_i P_\ell^{I,i}(\Delta_\ell(1+x)) &= [x]_{d_\ell}^T A_\ell^I [x]_{d_\ell} + x([x]_{d'_\ell}^T B_\ell^I [x]_{d'_\ell}) \quad \text{for } 0 \leq \ell \leq L, \\ A_\ell^I, B_\ell^I &\succeq 0 \quad \text{for } 0 \leq \ell \leq L. \end{aligned} \tag{4.119}$$

For brevity we here we use the index $i = 1, \dots, k(k+1)/2 \times \dim(V_{\Delta,\ell}^I)$ to run over all of the z and \bar{z} derivatives under consideration, as well as the components of the vector $V_{\Delta,\ell}^I$. I runs over possible global symmetry representations, and A_ℓ^I and B_ℓ^I are positive semidefinite matrices. We recall that $[x]_d$ is the vector with entries $(1, x, \dots, x^d)$, and if the polynomial $P_\ell^{I,i}$ has degree $2\gamma_\ell + 1 - \epsilon_\ell$ (with $\epsilon_\ell = 0, 1$), then $d_\ell = 2\gamma_\ell$ and $d'_\ell = 2\gamma_\ell - 2\epsilon_\ell$.

The middle constraint is an equality between polynomials in x , so in practice we will implement it by matching each polynomial coefficient:

$$0 = \mathbf{coeffs}_x \left[-a_i P_\ell^{I,i}(\Delta_\ell(1+x)) + \text{Tr}(X_{d_\ell}(x) A_\ell^I) + x \text{Tr}(X_{d'_\ell}(x) B_\ell^I) \right]. \tag{4.120}$$

In this expression we have also defined the matrix $X_d(x) \equiv [x]_d [x]_d^T$. Since many SDP solvers only allow positive variables, in practice it will additionally be convenient to introduce a ‘slack variable’ s , where without loss of generality we can replace $a_i \rightarrow a_i - s$ in the above expressions and require $a_i, s \geq 0$.

We solve the above semidefinite program using **SDPA-GMP 7.1.2** [117], which utilizes the GNU Multiple Precision Arithmetic Library (GMP). We use **Mathematica 7.0** to compute the vectors $V_\ell^{I,i}$ and polynomials $P_\ell^{I,i}$, performing all computations using 100 digits of precision. When using the approximations of Eqs. (4.106) and (4.116) we keep four roots, leading to approximations that differ from the exact functions by $\sim 10^{-8} - 10^{-10}$, depending on the value of β . In our computations we have found it sufficient to take $L = 20$; in addition we add constraints for $\ell = 1000, 1001$ in order to effectively include the asymptotic constraints at large ℓ . After setting up the problem in **Mathematica**, we write the SDP to a file using the SDPA sparse data format.

When running **SDPA-GMP**, we use the parameters:

SDPA-GMP Parameter	Value
maxIteration	1000
epsilonStar	10^{-10}
lambdaStar	10^{20}
omegaStar	10^{20}
lowerBound	-10^{20}
upperBound	10^{20}
betaStar	0.1
betaBar	0.3
gammaStar	0.9
epsilonDash	10^{-10}
precision	200

To make our plots, we run data points in parallel using the Odyssey computing cluster at Harvard University. In the majority of our plots we use a horizontal spacing of $\delta d = 10^{-2}$, supplemented by a higher resolution scan with $\delta d = 10^{-3}$ for $d < 1.01$ ($\delta d \rightarrow \delta \Delta_0$ in figure 4.10). To compute dimension bounds, we vary Δ_0 using a binary search, terminating at a vertical resolution of 10^{-3} . In all cases that we have checked, increasing L or including more roots in the polynomial approximation leads to a completely negligible ($\lesssim 10^{-4}$) change in the computed bound.

Chapter 5

Bootstrapping the 3D Ising Model

This chapter is a lightly-edited version of

S. El-Showk, M. F. Paulos, D. Poland, S. Rychkov, D. Simmons-Duffin and A. Vichi,
“Solving the 3D Ising Model with the Conformal Bootstrap,” arXiv:1203.6064 [hep-th].

5.1 Introduction

This paper is the first in a series of works which will conceivably lead to a solution of the Conformal Field Theory (CFT) describing the three dimensional (3D) Ising model at the critical temperature. Second-order phase transitions in a number of real-world systems are known to belong to the same universality class: most notably liquid-vapor transitions and transitions in binary fluids and uniaxial magnets.

Field-theoretical descriptions of critical phenomena and computations of critical exponents have a long tradition [118]. One well-known approach to this problem is the ϵ -expansion [119]. In this method the critical exponents are computed using the usual, perturbative field theory in $D = 4 - \epsilon$ dimensions, and the physically interesting case of $D = 3$ is obtained by extrapolating to $\epsilon = 1$. The obtained series in ϵ are divergent and need to be resummed. Apart from small

ambiguities, the final results for the critical exponents agree well with experiments and with a host of other approximation techniques (high-temperature expansion, Monte-Carlo simulations, etc.).

In this paper we will develop an alternative method for determining critical exponents in $D = 3$, based on Polyakov's hypothesis of conformal invariance of critical fluctuations [99], which was a major motivation for the development of Conformal Field Theory. CFT methods have been extremely fruitful in $D = 2$, allowing one to solve many models of critical behavior [120]. The novelty of our project is to apply them in $D = 3$. The existing quantitative approaches to critical phenomena in $D = 3$ do not take full advantage of conformal invariance.

The CFT describing the 3D Ising model at criticality is not known to possess any additional symmetry apart from conformal invariance and \mathbb{Z}_2 invariance. For this reason we will be able to rely only on the most general properties of conformal theories. The study of such general properties goes back to the 1970s. The required fundamental concepts are the classification of primary operators, the conformally-invariant operator product expansion, conformal blocks, and the idea of the nonperturbative conformal bootstrap, which were introduced in the work of Mack and Salam [47], Ferrara, Gatto, Grillo and Parisi [51, 101, 121–124] and Polyakov [52]. In addition, we will need explicit expressions for the conformal blocks. Here we will be able to rely on the recent work of Dolan and Osborn [64, 65, 125].

While most of these ingredients were understood many years ago, until recently it was not known how to put them together in order to extract dynamical information about CFTs. This important know-how was developed in a series of recent papers [54–61]. That work was largely motivated by particle physics (in particular the theory of electroweak symmetry breaking) and concerned CFT in $D = 4$. However, the time is now ripe to transfer these techniques to $D = 3$. The cases $D = 3$ and $D = 4$ are similar in that the conformal algebra has finitely many generators (unlike in $D = 2$ where it has an infinite-dimensional extension, the Virasoro algebra).

This paper is structured as follows. In Section 5.2 we review what is known about the

operator content of the 3D Ising model. In Section 5.3 we discuss the conformal bootstrap approach to studying 3D CFTs, and in Section 5.4 we present an efficient method for computing the conformal partial waves appearing in four-point functions of scalars for CFTs in any dimension (including $D = 3$). In Section 5.5 we present bounds on 3D CFTs that follow from crossing symmetry and compare them to what is known about the 3D Ising model. Finally, we discuss our results and future directions for this program in Section 5.6.

5.2 Operator Content of the 3D Ising Model

We assume that the reader is familiar with the basic facts about the Ising model and the critical phenomena in general, see [118, 126–128].

In this paper, we will be aiming for a solution of the 3D Ising model *in the continuum limit* and *at the critical temperature* $T = T_c$. While the 2D Ising model was solved exactly *on the lattice* and *for any temperature* by Onsager and Kaufman in the 1940's, the 3D lattice case has resisted all attempts for an exact solution. Istrail [129] proved in 2000 that solving the 3D Ising model on the lattice is an NP-complete problem. However, this theorem does not exclude the possibility of finding a solution in the continuum limit.

The standard way to think about the continuum theory is in terms of local operators (or fields). At $T = T_c$, the theory has scale (and, as we discuss below, conformal) invariance, and each operator is characterized by its scaling dimension Δ and $O(3)$ spin. The operators of spin higher than 1 are traceless symmetric tensors.

In Table 5.1 we list a few notable local operators, which split into odd and even sectors under the global \mathbb{Z}_2 symmetry (the Ising spin flip). The operators σ and ε are the lowest dimension \mathbb{Z}_2 -odd and even scalars respectively—these are the continuum space versions of the Ising spin and of the product of two neighboring spins on the lattice. The two next-to-lowest scalars in each

\mathbb{Z}_2 -sector are called σ' and ε' . Their dimensions are related to the irrelevant critical exponents ω_A and ω measuring corrections to scaling. The operator ε'' is analogously related to the next-to-leading \mathbb{Z}_2 -even irrelevant exponent ω_2 . The stress tensor $T_{\mu\nu}$ has spin 2 and, as a consequence of being conserved, canonical dimension $\Delta_T = 3$. The lowest-dimension spin 4 operator $C_{\mu\nu\kappa\lambda}$ has a small anomalous dimension, related to the critical exponent ω_{NR} measuring effects of rotational symmetry breaking on the cubic lattice.

Operator	Spin l	\mathbb{Z}_2	Δ	Exponent
σ	0	−	0.5182(3)	$\Delta = 1/2 + \eta/2$
σ'	0	−	$\gtrsim 4.5$	$\Delta = 3 + \omega_A$
ε	0	+	1.413(1)	$\Delta = 3 - 1/\nu$
ε'	0	+	3.84(4)	$\Delta = 3 + \omega$
ε''	0	+	4.67(11)	$\Delta = 3 + \omega_2$
$T_{\mu\nu}$	2	+	3	n/a
$C_{\mu\nu\kappa\lambda}$	4	+	5.0208(12)	$\Delta = 3 + \omega_{\text{NR}}$

Table 5.1: Notable low-lying operators of the 3D Ising model at criticality.

The approximate values of operator dimensions given in the table have been determined from a variety of theoretical techniques, most notably the ϵ -expansion, high temperature expansion, and Monte-Carlo simulations; see p. 47 of Ref. [118] for a summary. The achieved precision is rather impressive for the lowest operator in each class, but quickly gets worse for the higher fields. While ultimately we would like to beat the old methods, it would be unwise to completely dismiss this known information and restart from scratch. Rather, we will be using it for guidance while sharpening our own methods.

Among the old techniques, the ϵ -expansion of Wilson and Fisher [119] deserves a separate comment. The well-known idea of this approach is that the 3D Ising critical point and the 4D free scalar theory can be connected by a line of fixed points by allowing the dimension of space to vary continuously between 3 and 4. For $D = 4 - \epsilon$, the Wilson-Fisher fixed point is weakly coupled and the dimensions of local operators can be expanded order-by-order in ϵ . For the most

important operators, like σ and ε , these expansions have been extended to terms of order as high as ϵ^5 [128], requiring a five-loop perturbative field theory computation. However, as often happens in perturbation theory, the resulting series are only asymptotic. For the physically interesting case $\epsilon = 1$, their divergent nature already starts to show after the first couple of terms. Nevertheless, after appropriate resummation the ϵ -expansion produces results in agreement with the other methods. So its basic hypothesis must be right, and can give useful qualitative information about the 3D Ising operator spectrum, even where accurate quantitative computations are missing.

It is now time to bring up the conformal invariance of the critical point, conjectured by Polyakov [99]. This symmetry is left unused in the RG calculations leading to the ϵ -expansion, and in most other existing techniques.¹ This is because it only emerges at the critical point; it's not present along the flow. Conformal invariance seems to be a generic feature of criticality, but why exactly is not fully understood [1]. Recently there has been a renewed interest in the question of whether there exist interesting scale invariant but not conformal systems [132–137]. We will simply assume as a working hypothesis that the 3D Ising critical point is conformal.

A nice experimental test of conformal invariance would be to measure the three-point function $\langle \sigma(x)\sigma(y)\varepsilon(z) \rangle$ on the lattice, to see if its functional form agrees with the one fixed by conformal symmetry [99]. We do not know if this has been done.

Using 3D conformal invariance, local operators can be classified into primaries and descendants [47]. The primaries² transform homogeneously under the finite-dimensional conformal group, while the descendants are derivatives of primaries and transform accordingly. All operators listed in Table 5.1 are primaries. This is obvious for σ and ε —the lowest dimension scalars in each \mathbb{Z}_2 -symmetry class. That $\sigma', \varepsilon', \varepsilon'', C_{\mu\nu\kappa\lambda}$ are all primaries and not derivative operators follows from the fact that they are associated with corrections to scaling, while adding a derivative operator to

¹Conformal invariance has been used in studies of critical $O(N)$ models in the large N limit [130, 131].

²These are usually called quasi-primaries in 2D CFTs.

the Lagrangian has no effect. Finally, the stress tensor is always a primary.

It can be seen that all operators in Table 5.1 have non-negative anomalous dimensions (by which we mean the difference between the operator dimension and the dimension of the lowest 3D free scalar theory operator with the same quantum numbers). This is not accidental, but is related to reflection positivity, which is the Euclidean space version of unitarity. Primaries in reflection positive (or unitary) CFTs are known to have non-negative anomalous dimensions [94, 100, 101, 138, 139]:

$$\Delta \geq D/2 - 1 \quad (l = 0), \quad \Delta \geq l + D - 2 \quad (l \geq 1). \quad (5.1)$$

The 3D Ising model is reflection positive on the lattice [140], and this property is inherited in the continuum limit, so that the ‘unitarity bounds’ (5.1) are respected.

Can conformal symmetry be used to *determine* the local operator dimensions rather than to interpret the results obtained via other techniques? In 2D this was done long ago [120] using the Virasoro algebra. This also justified *post factum* the assumption of conformal invariance, since the critical exponents and other quantities agreed with the exact lattice solution. The Virasoro algebra does not extend to 3D, but in the next section we will describe a method which is applicable for any D .

5.3 Conformal Bootstrap

Primary operators in a CFT form an algebra under the Operator Product Expansion (OPE). This means that the product of two primary operators at nearby points can be replaced inside a correlation function by a series in other local operators times coordinate-dependent coefficient functions. Schematically, the OPE of two primaries has the form:

$$\phi_i(x_1)\phi_j(x_2) = \sum_k f_{ijk} C(x_1 - x_2, \partial_2)\phi_k(x_2). \quad (5.2)$$

The differential operators C are fixed by conformal invariance, and only primary operators need to be included in the sum on the RHS. Here we are suppressing indices for clarity. In general, scalar operators as well as operators of nonzero spin will appear on the RHS. Fairly explicit expressions for the C 's have been known since the 70's, at least in the case when the $\phi_{i,j}$ are scalars and ϕ_k is a traceless symmetric tensor of arbitrary rank [64, 121], but we will not need them here.

The numerical coefficients f_{ijk} are called structure constants, or OPE coefficients. These numbers, along with the dimensions and spins of all primary fields, comprise the ‘CFT data’ characterizing the algebra of local operators.

The conformal bootstrap condition [52, 120, 122], shown schematically in Fig. 5.1, says that the operator algebra must be associative. In that figure we consider the correlator of four primaries

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4) \rangle \quad (5.3)$$

and use the OPE in the (12)(34) or (14)(23)-channel to reduce it to a sum of two-point functions. The answer should be the same, which gives a quadratic condition on the structure constants of the schematic form

$$\sum_k f_{12k} f_{34k}(\dots) = \sum_k f_{14k} f_{23k}(\dots). \quad (5.4)$$

The (\dots) factors are functions of coordinates x_i , called conformal partial waves. They are produced by acting on the two-point function of the exchanged primary field ϕ_k with the differential operators C appearing in the OPE of two external primaries. Thus, they are also fixed by conformal invariance in terms of the dimensions and spins of the involved fields.

The dream of the conformal bootstrap is that the condition (5.4), when imposed on four-point functions of sufficiently many (all?) primary fields, should allow one to determine the CFT data and thus solve the CFT. Of course, there are presumably many different CFTs, and so one can expect some (discrete?) set of solutions. One of the criteria which will help us to select the solution representing the 3D Ising model is the global symmetry group, which must be \mathbb{Z}_2 .

$$\sum_k \text{Diagram 1} = \sum_k \text{Diagram 2}$$

Figure 5.1: The conformal bootstrap condition = associativity of the operator algebra.

Our method of dealing with the conformal bootstrap will require explicit knowledge of the conformal partial waves. In the next section we will gather the needed results.

5.4 Conformal Blocks

In this paper we will be imposing the bootstrap condition only on four-point functions of scalars. Conformal partial waves for such correlators were introduced in [51] and further studied in [123, 124]; they were also discussed in [52]. Recently, new deep results about them were obtained in [64, 65, 125]. Significant progress in understanding non-scalar conformal partial waves was made recently in [68] (building on [69]), which also contains a concise introduction to the concept. Below we'll normalize the scalar conformal partial waves as in [125]; see Appendix 5.A for further details on our conventions.

Consider a correlation function of four scalar primaries ϕ_i of dimension Δ_i , which is fixed by conformal invariance to have the form [99]

$$\langle \phi_1(x_1) \phi_2(x_2) \phi_3(x_3) \phi_4(x_4) \rangle = \left(\frac{x_{24}^2}{x_{14}^2} \right)^{\frac{1}{2}\Delta_{12}} \left(\frac{x_{14}^2}{x_{13}^2} \right)^{\frac{1}{2}\Delta_{34}} \frac{g(u, v)}{(x_{12}^2)^{\frac{1}{2}(\Delta_1+\Delta_2)} (x_{34}^2)^{\frac{1}{2}(\Delta_3+\Delta_4)}}, \quad (5.5)$$

where $x_{ij} \equiv x_i - x_j$, $\Delta_{ij} \equiv \Delta_i - \Delta_j$, and $g(u, v)$ is a function of the conformally invariant cross-ratios

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \quad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}. \quad (5.6)$$

The conformal partial wave expansion in the (12)(34) channel gives a series representation for this function:

$$g(u, v) = \sum_{\mathcal{O}} f_{12\mathcal{O}} f_{34\mathcal{O}} G_{\Delta, l}(u, v), \quad (5.7)$$

where the sum is over the exchanged primaries \mathcal{O} of dimension Δ and spin l and the functions $G_{\Delta, l}(u, v)$ are called conformal blocks. We must learn to compute them efficiently.

In even dimensions, conformal blocks have relatively simple closed-form expressions in terms of hypergeometric functions [64, 65, 124, 125]. For example, the 2D and 4D blocks are given by:

$$\begin{aligned} G_{\Delta, l}^{D=2}(u, v) &= \frac{1}{2} [k_{\Delta+l}(z)k_{\Delta-l}(\bar{z}) + (z \leftrightarrow \bar{z})], \\ G_{\Delta, l}^{D=4}(u, v) &= \frac{1}{l+1} \frac{z\bar{z}}{z-\bar{z}} [k_{\Delta+l}(z)k_{\Delta-l-2}(\bar{z}) - (z \leftrightarrow \bar{z})], \end{aligned} \quad (5.8)$$

where

$$k_{\beta}(x) \equiv x^{\beta/2} {}_2F_1\left(\frac{1}{2}(\beta - \Delta_{12}), \frac{1}{2}(\beta + \Delta_{34}); \beta; x\right), \quad (5.9)$$

and the complex variable z and its complex conjugate \bar{z} are related to u, v via

$$u = z\bar{z}, \quad v = (1-z)(1-\bar{z}). \quad (5.10)$$

The meaning of the variable z is explained in Fig. 5.2. From the known analyticity properties of ${}_2F_1$, it follows that the conformal blocks are smooth single-valued functions in the z plane minus the origin and the $(1, +\infty)$ cut along the real axis. This is not accidental and should be valid for any D . By standard radial quantization reasoning (see [141], Sec. 2.9), the OPE by which the conformal blocks are defined is expected to converge as long as there is a sphere separating x_1 and x_2 from x_3 and x_4 . This sphere degenerates into a plane and disappears precisely when z crosses the cut.

We now pass to the results for general D , including the case $D = 3$ we are interested in, which are rather more complicated. From now on we consider only conformal blocks of four

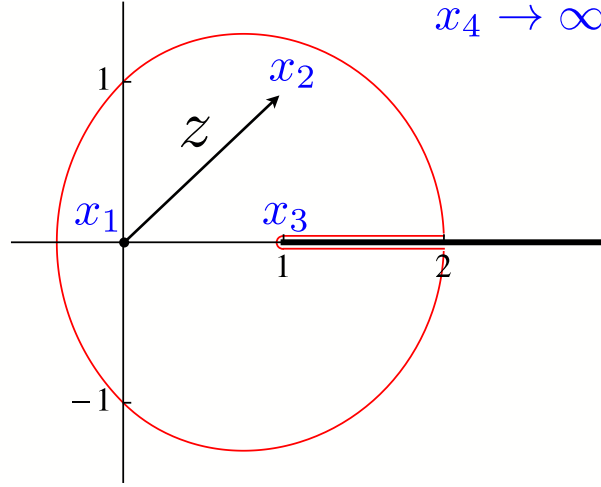


Figure 5.2: Using conformal freedom, three operators can be fixed at $x_1 = 0$, $x_3 = (1, 0, \dots, 0)$, $x_4 \rightarrow \infty$, while the fourth point x_2 can be assumed to lie in the (12) plane. The variable z is then the complex coordinate of x_2 in this plane, while \bar{z} is its complex conjugate. Also shown: the conformal block analyticity cut (thick black), and the boundary of the absolute convergence region of the power series representation (5.11) (thin red).

identical scalars, so that $\Delta_{12} = \Delta_{34} = 0$. In any dimension, such blocks depend only on the dimension and spin of the exchanged primary. For scalar exchange ($l = 0$), conformal blocks have a double power series representation ([64], Eq. (2.32)):

$$G_{\Delta,0}(u, v) = u^{\Delta/2} \sum_{m,n=0}^{\infty} \frac{[(\Delta/2)_m (\Delta/2)_{m+n}]^2}{m! n! (\Delta + 1 - \frac{D}{2})_m (\Delta)_{2m+n}} u^m (1-v)^n, \quad (5.11)$$

where $(x)_n$ is the Pochhammer symbol. In this paper we will only use this representation at $z = \bar{z}$, in order to derive the closed form expression (5.14) given below. In principle the series converges absolutely in the region

$$|1-v| < \begin{cases} 1, & 0 \leq u < 1, \\ 2\sqrt{u} - u, & u \geq 1, \end{cases} \quad (5.12)$$

whose boundary is traced in red in Fig. 5.2.

For exchanged operators of nonzero spin, the conformal blocks can be computed via various recursion relations. Some recursion relations previously appeared in [64], Eqs. (2.30), but these will

not be useful for us since they express the blocks with equal external dimensions in terms of blocks where the external dimensions differ by an integer.

In Appendix 5.A, we exhibit a recursion relation which follows from the results of Ref. [125] and does not require shifts in the external dimensions. In general, this recursion involves taking derivatives of $G_{\Delta,l}(z, \bar{z})$, which is not very easy to perform numerically. However, along the line $z = \bar{z}$ the terms involving derivatives drop out and the recursion relation for the conformal block $G_{\Delta,l}(z) \equiv G_{\Delta,l}(z, z)$ becomes extremely simple:

$$\begin{aligned} (l + D - 3)(2\Delta + 2 - D)G_{\Delta,l}(z) \\ = (D - 2)(\Delta + l - 1)G_{\Delta,l-2}(z) + \frac{2 - z}{2z} (2l + D - 4)(\Delta - D + 2)G_{\Delta+1,l-1}(z) \\ - \frac{\Delta(2l + D - 4)(\Delta + 2 - D)(\Delta + 3 - D)(\Delta - l - D + 4)^2}{16(\Delta + 1 - \frac{D}{2})(\Delta - \frac{D}{2} + 2)(l - \Delta + D - 5)(l - \Delta + D - 3)}G_{\Delta+2,l-2}(z). \end{aligned} \quad (5.13)$$

This recursion relation can easily compute all conformal blocks along the $z = \bar{z}$ line in terms of spin 0 and 1 blocks.³ On the other hand, as shown in Appendix 5.B, the spin 0 and 1 blocks along the $z = \bar{z}$ line can be simply expressed in terms of generalized hypergeometric functions ($\alpha \equiv D/2 - 1$):

$$G_{\Delta,0}(z) = \left(\frac{z^2}{1 - z} \right)^{\Delta/2} {}_3F_2 \left(\frac{\Delta}{2}, \frac{\Delta}{2}, \frac{\Delta}{2} - \alpha; \frac{\Delta+1}{2}, \Delta - \alpha; \frac{z^2}{4(z-1)} \right), \quad (5.14)$$

$$G_{\Delta,1}(z) = \frac{2 - z}{2z} \left(\frac{z^2}{1 - z} \right)^{\frac{\Delta+1}{2}} {}_3F_2 \left(\frac{\Delta+1}{2}, \frac{\Delta+1}{2}, \frac{\Delta+1}{2} - \alpha; \frac{\Delta}{2} + 1, \Delta - \alpha; \frac{z^2}{4(z-1)} \right). \quad (5.15)$$

These explicit expressions, together with the recursion relation (5.13), solve the problem of finding conformal blocks along the $z = \bar{z}$ line. What about $z \neq \bar{z}$? We should explain that in our numerical implementation of conformal bootstrap we will not actually use the values of conformal blocks at generic values of z . Instead, we will Taylor-expand the conformal bootstrap condition around the point $z = \bar{z} = 1/2$. This is an approach which proved efficient in prior work in 4D and

³This works for general D . In $D = 3$, one can instead recurse from $G_{\Delta,0}$ and $G_{\Delta,-1} \equiv G_{\Delta,0}$, where the latter equality follows from (5.38).

2D, and we will pursue it here as well. So, we will have to evaluate derivatives of conformal blocks at the point $z = \bar{z} = 1/2$, both along and transverse to the $z = \bar{z}$ line.

Now, derivatives along the $z = \bar{z}$ line will be evaluated as follows. For the spin 0 and 1 conformal blocks we can take advantage of the fact that the ${}_3F_2$ hypergeometric functions satisfy a third-order differential equation:

$$\left(x \hat{D}_{a_1} \hat{D}_{a_2} \hat{D}_{a_3} - \hat{D}_0 \hat{D}_{b_1-1} \hat{D}_{b_2-1} \right) {}_3F_2(a_1, a_2, a_3; b_1, b_2; x) = 0, \quad (5.16)$$

where $\hat{D}_c \equiv x \partial_x + c$. This equation can be used to obtain recursion relations which express the third-order and higher derivatives of the spin 0 and 1 blocks in terms of their first and second derivatives. The values of the latter derivatives at $z = \bar{z} = 1/2$ will be tabulated as a function of Δ .

Derivatives of the higher spin blocks are then computed using the recursion relations following from (5.13). This completely settles the question of obtaining derivatives along the $z = \bar{z}$ line.

In order to obtain the derivatives transverse to the $z = \bar{z}$ line, we'll take advantage of the fact that conformal partial waves are eigenfunctions of the quadratic Casimir operator of the conformal group, which implies that conformal blocks satisfy a second-order differential equation [65]:

$$\mathcal{D} G_{\Delta, l}(z, \bar{z}) = \frac{1}{2} C_{\Delta, l} G_{\Delta, l}(z, \bar{z}), \quad (5.17)$$

where $C_{\Delta, l} \equiv \Delta(\Delta - D) + l(l + D - 2)$ and

$$\mathcal{D} \equiv (1 - z) z^2 \partial_z^2 - \left[z^2 - (D - 2) \frac{z \bar{z} (1 - z)}{z - \bar{z}} \right] \partial_z + (z \leftrightarrow \bar{z}). \quad (5.18)$$

Let us now make a change of variables:

$$z = (a + \sqrt{b})/2, \quad \bar{z} = (a - \sqrt{b})/2. \quad (5.19)$$

The point $z = \bar{z} = 1/2$ which interests us corresponds to $a = 1$, $b = 0$. Moreover, since conformal blocks are symmetric in $z \leftrightarrow \bar{z}$, their power series expansion away from the $z = \bar{z}$ line will contain only even powers of $(z - \bar{z})$, and hence integer powers of b . In the new variables the differential operator \mathcal{D} takes the form

$$\begin{aligned} \mathcal{D} = & (2-a)a^2 \left[\frac{1}{2}(D-1)\partial_b + b\partial_b^2 \right] \\ & + (2-3a)b^2\partial_b^2 + \left[\frac{1}{2}(D-9)a - a(3a-4)\partial_a - D + 3 \right] b\partial_b - \frac{1}{4}Da^2\partial_a + \frac{1}{4}(2-a)a^2\partial_a^2 \\ & - b^2\partial_a\partial_b + \left[\frac{1}{4}(D-4)\partial_a + \frac{1}{4}(2-3a)\partial_a^2 \right] b, \end{aligned} \quad (5.20)$$

where the terms have been grouped into lines according to how they change the power of b . The first line contains the *leading terms*, which lower the power of b by one unit. Notice that the leading terms generate a nonvanishing coefficient when acting on any positive power of b , as long as $0 < a < 2$ (which corresponds to $0 < z < 1$). Thus, in a neighborhood of this interval the Casimir differential equation (5.17) can be solved *à la* Cauchy-Kovalevskaya, recursively in a power series expansion in b using the known conformal blocks at $b = 0$ as a boundary value.

Let us denote the $\partial_a^m \partial_b^n$ derivative of the conformal block evaluated at $z = \bar{z} = 1/2$ by $h_{m,n}$. Since we know the conformal blocks along the $z = \bar{z}$ line, we can compute all the derivatives $h_{m,0}$. On the other hand, the Cauchy-Kovalevskaya argument above implies that there will be a recursion relation for $h_{m,n}$ (with $n > 0$) in terms of $h_{m,n}$ with lower values of n . The recursion relation is given in Appendix 5.C and has the general structure:

$$h_{m,n} = \sum_{m' \leq m-1} m(\dots)h_{m',n} + \sum_{m' \leq m+2} [(\dots)h_{m',n-1} + (n-1)(\dots)h_{m',n-2}]. \quad (5.21)$$

The appearance of m' up to $m+2$ is related to the fact that, the Casimir equation being second-order, derivatives of up to second-order in a appear in the RHS of (5.20). The first term being proportional to m ensures that $h_{0,n}$ terms generated by repeatedly applying the recursion are eventually reduced to $h_{1,0}$ and $h_{0,0}$. This recursion then solves the problem of computing the

conformal block derivatives transverse to the line $z = \bar{z}$.

5.5 Bounds and Consequences for the 3D Ising Model

In this section we will use the bootstrap equations discussed above in order to derive rigorous bounds on 3D CFTs. When comparing these bounds to the 3D Ising model, we'll focus on constraints coming from the four-point function of the Ising spin operator $\langle \sigma \sigma \sigma \sigma \rangle$. The conformal block expansion of this four-point function has the form

$$g(u, v) = \sum p_{\Delta, l} G_{\Delta, l}(u, v), \quad p_{\Delta, l} \equiv f_{\Delta, l}^2 \geq 0, \quad (5.22)$$

where the sum runs over the dimensions and spins of all primary operators appearing in the $\sigma \times \sigma$ OPE. This OPE contains all of the \mathbb{Z}_2 -even operators listed in Table 5.1, in addition to infinitely many other even-spin operators. Note that odd-spin operators cannot appear because of Bose symmetry. The coefficients $p_{\Delta, l}$ appearing in the conformal block expansion are squares of the OPE coefficients, and are thus constrained to be positive.

The conformal bootstrap equation (5.4) takes a particularly simple form for this correlator, since the (12)(34) and (14)(23) channel involve the same OPE coefficients. It can be stated as a crossing symmetry constraint on the function $g(u, v)$:

$$v^{\Delta_\sigma} g(u, v) = u^{\Delta_\sigma} g(v, u). \quad (5.23)$$

Substituting the conformal block decomposition, we get an equation

$$u^{\Delta_\sigma} - v^{\Delta_\sigma} = \sum' p_{\Delta, l} [v^{\Delta_\sigma} G_{\Delta, l}(u, v) - u^{\Delta_\sigma} G_{\Delta, l}(v, u)], \quad (5.24)$$

where \sum' is the sum over all operators except the unit operator, whose contribution has been separated in the LHS. It was shown in [54] and confirmed in subsequent work [55–61] that this type of equation can be used to extract dynamical information about 4D and 2D CFTs. We will now apply the same methods in 3D.

First, we will Taylor-expand (5.24) around the point $z = \bar{z} = 1/2$ up to some large fixed order. That this is a reasonable point to expand around follows from the fact that it is democratic with respect to the direct and crossed channels in the conformal block decomposition: by making a conformal transformation the four points can be put at the vertices of a square.

The Taylor-expanded (5.24) can be viewed as a finite system of linear equations (one for each Taylor coefficient) for a large (strictly speaking infinite) number of variables $p_{\Delta,l}$. A priori, there is one variable $p_{\Delta,l}$ for each pair (Δ, l) consistent with the unitarity bounds (5.1). However, one may wish to posit additional constraints on the spectrum (such as assumptions about gaps). Below we will study which of these constraints are consistent with the existence of a solution.

This system of linear equations should also be augmented by inequalities expressing the fact that variables $p_{\Delta,l}$ are non-negative. Fortunately, problems involving linear inequalities are almost as tractable as pure systems of linear equalities. These problems form a chapter of linear algebra called linear programming, and there exist efficient algorithms for solving them (such as Dantzig's simplex method or interior point methods). Once the additional constraints on the spectrum are specified, one can use linear programming methods to find out if the system has a solution. If the answer is negative, a CFT with such a spectrum cannot exist. The details of our implementation of this problem are given in Appendix 5.D.

5.5.1 Bounds on Δ_ε

We are now ready to start asking concrete questions about the 3D Ising CFT to which we can give unambiguous answers. The first question is as follows. Let's be agnostic about the dimension of the spin field, allowing it to vary in the interval $0.5 \leq \Delta_\sigma \lesssim 0.8$. The lower end of this interval is fixed by the unitarity bound, while the upper end has been chosen arbitrarily. For each Δ_σ in this range, we ask: *What is the maximal Δ_ε allowed by (5.24)?*

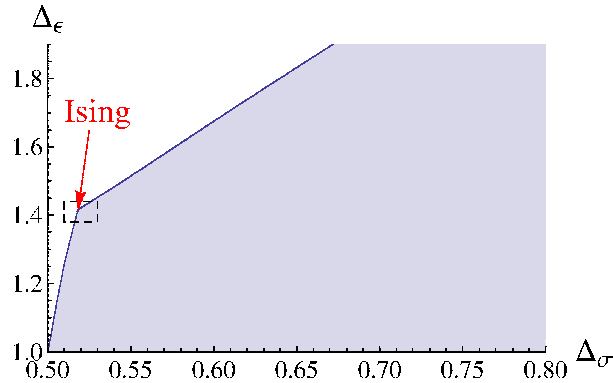


Figure 5.3: Shaded: the part of the $(\Delta_\sigma, \Delta_\epsilon)$ plane allowed by the crossing symmetry constraint (5.24). The boundary of this region has a kink remarkably close to the known 3D Ising model operator dimensions (the tip of the arrow). The zoom of the dashed rectangle area is shown in Fig. 5.4. This plot was obtained with the algorithm described in Appendix 5.D with $n_{\max} = 11$.

The result is plotted in Fig. 5.3: only the points $(\Delta_\sigma, \Delta_\epsilon)$ in the shaded region are allowed.⁴ Just like similar plots in 4D and 2D [54, 55, 61] the curve bounding the allowed region starts at the free theory point and rises steadily. Moreover, just like in 2D [55] the curve shows a kink whose position looks remarkably close to the Ising model point.⁵ This is better seen in Fig. 5.4 where we zoom in on the kink region. The boundary of the allowed region intersects the red rectangle drawn using the Δ_σ and Δ_ϵ error bands given in Table 5.1.

From this comparison, we can draw two solid conclusions. First of all, the old results for the allowed dimensions are not inconsistent with conformal invariance, though they are based on completely different techniques. Second, we can rigorously rule out about half of the $(\Delta_\sigma, \Delta_\epsilon)$ rectangle allowed by the table. It seems that the 3D Ising model lies remarkably close to the boundary of the allowed region, if not on the boundary. At present we don't have an explanation of why this had to be the case.

⁴To avoid possible confusion: we show only the upper boundary of the allowed region. $0.5 \leq \Delta_\epsilon \leq 1$ is also a priori allowed.

⁵In contrast, the 4D dimension bounds do not show kinks, except in supersymmetric theories [61].

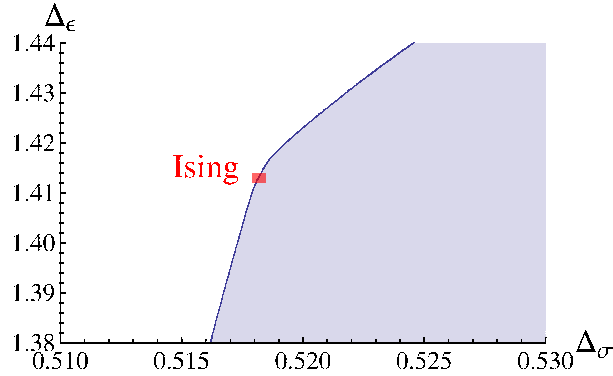


Figure 5.4: The zoom of the dashed rectangle area from Fig. 5.3. The small red rectangle is drawn using the Δ_σ and Δ_ε error bands given in Table 5.1.

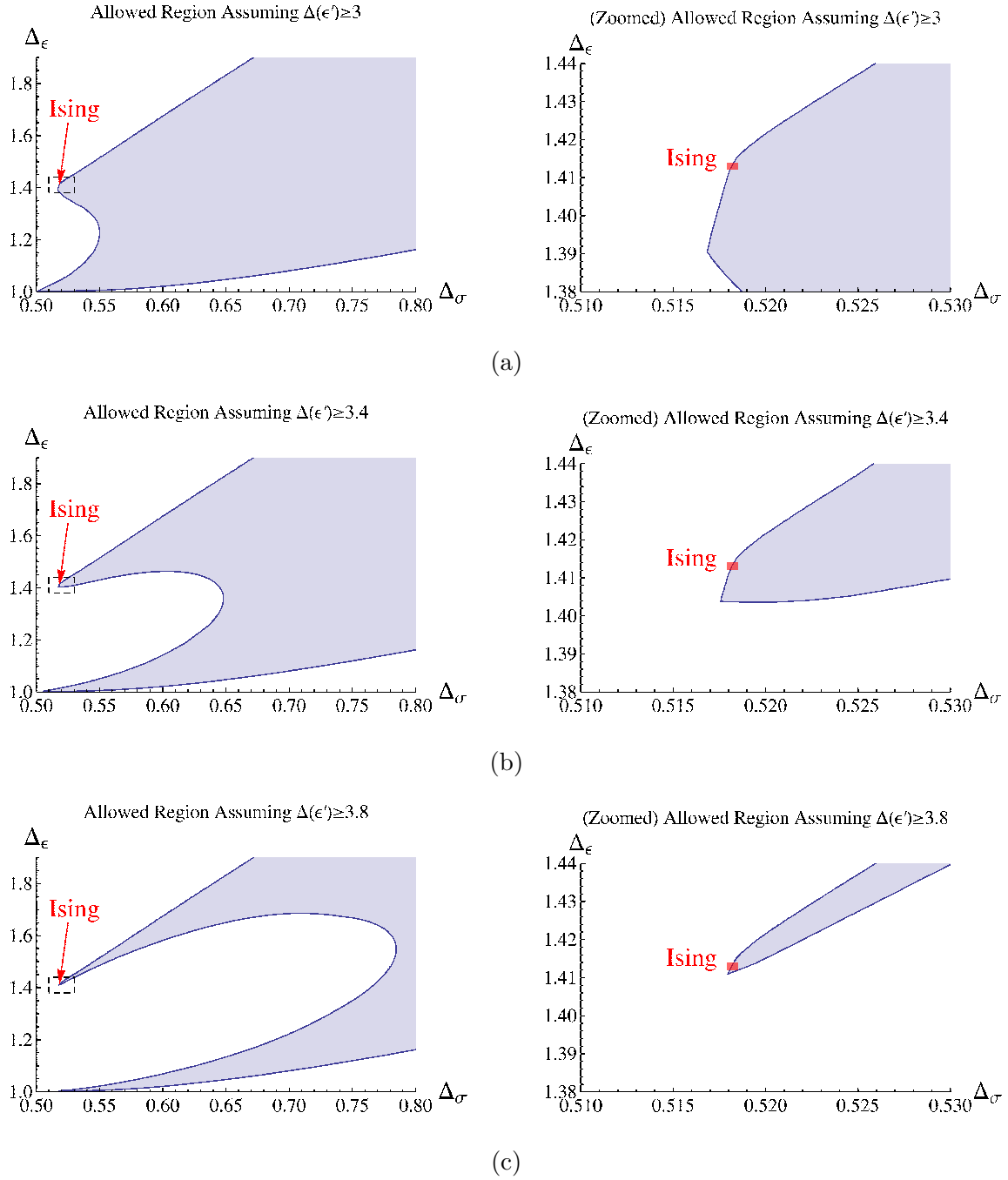
5.5.2 Bounds Assuming a Gap Between ε and ε'

We will next give a series of plots showing the impact of assuming a gap in the \mathbb{Z}_2 -even scalar spectrum (as proposed in [62]). In other words, we will impose that the first operator after ε has dimension $\Delta_{\varepsilon'}$ above a certain value.

Going from weaker to stronger, we will consider three constraints: $\Delta_{\varepsilon'} \geq 3, 3.4, 3.8$. Thus, we will ask: *What is the region of the $(\Delta_\sigma, \Delta_\varepsilon)$ plane allowed by (5.24) when this extra constraint is taken into account?*

The weakest of the three assumptions, $\Delta_{\varepsilon'} \geq 3$, has been chosen since it can be justified experimentally: we know that the 3D Ising critical point is reached by fine-tuning just one parameter (the temperature). Therefore, it has just one relevant \mathbb{Z}_2 -even scalar, ε , while $\varepsilon', \varepsilon''$ etc. must be irrelevant. As we see in Fig. 5.5(a), this piece of information allows to exclude a fair part of the region allowed by Fig. 5.3. Unfortunately, close to the 3D Ising we do not gain constraining power: the new and the old bounds coincide there.

On the other hand, the stronger assumptions $\Delta_{\varepsilon'} \geq 3.4, 3.8$ exclude a much larger portion of dimension space, carving out an allowed region with two branches; see Figs. 5.5(b,c). The upper branch seems to end at the 3D Ising point, while the lower branch terminates near the free theory.

Figure 5.5: Same as Figs. 5.3, 5.4, but imposing the extra constraints $\Delta_{\epsilon'} \geq \{3, 3.4, 3.8\}$.

It is simple to understand why the intermediate region should not be allowed – assuming a gap $\Delta_{\varepsilon'} > \Delta_*$ should exclude the gaussian line $\Delta_\varepsilon = 2\Delta_\sigma$ up to a dimension of $\Delta_\sigma = \Delta_*/2 - 1$, since the spectrum of this solution is $2\Delta_\sigma + 2n + l$ for integer n . Our bounds are slightly weaker than that.

Zooming in on the tip near the 3D Ising point, we see that the allowed region in Fig. 5.5(c) barely intersects with the red rectangle. Were we to assume even larger gaps, the intersection would eventually disappear altogether. We performed this analysis and found that this happens for $\Delta_{\varepsilon'} \geq 3.840(2)$. This result rules out the upper half of the $\Delta_{\varepsilon'}$ range allowed by Table 5.1, assuming that the more accurate determinations of Δ_σ and Δ_ε in the same table are correct.

The same phenomenon is seen in a slightly different way in Fig. 5.6. Here we compute the maximal allowed $\Delta_{\varepsilon'}$ under the condition that Δ_ε has already been fixed to the maximal value allowed by Fig. 5.3. Notice the rapid growth of the $\Delta_{\varepsilon'}$ bound just below the 3D Ising model σ dimension, which allows ε' to become irrelevant. Similar growth has been observed in the 2D case in [62]. Around the 3D Ising Δ_σ the bound is $\Delta_{\varepsilon'} \lesssim 3.84$, consistent with the value cited above.

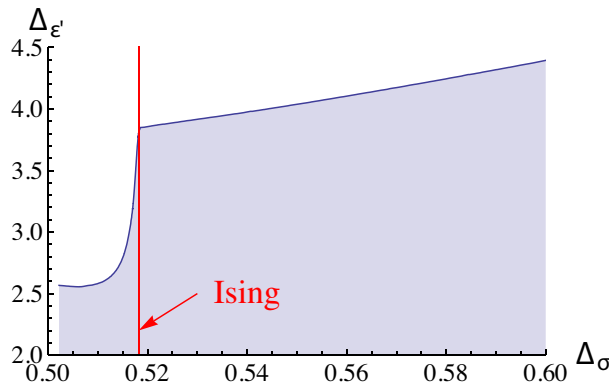


Figure 5.6: The bound on $\Delta_{\varepsilon'}$ under the condition that Δ_ε has already been fixed to the maximal value allowed by Fig. 5.3. Here $n_{\max} = 10$ (see Appendix 5.D). The width of the vertical red line marking the 3D Ising value of Δ_σ is about five times the error band in Table 5.1.

This story illustrates how the conformal bootstrap equation imposes nontrivial dependencies between various operator dimensions. Once some dimensions are determined, the other ones

are no longer arbitrary. Such interrelations are probably not easy to see from the renormalization group point of view. For instance, when using the ϵ -expansion, each of the operator dimensions listed in Table 5.1 requires an independent computation.

5.5.3 Bounds on the Gap in the Spin 2 Sector

The above discussion concerned the scalar sector of the 3D Ising model, but eventually we would like to also constrain operators with nonvanishing spin. For a first try, let's study here the gap in the spin 2 sector. The first spin 2 operator in the $\sigma \times \sigma$ OPE is the stress tensor $T_{\mu\nu}$, and we will be interested in the dimension of the second one, call it $T'_{\mu\nu}$.

In Fig. 5.7 we give a rigorous upper bound on $\Delta_{T'}$ following from the crossing symmetry constraint (5.24). The bound is shown as a function of Δ_σ only, and is in this sense analogous to our first bound in Fig. 5.3. Unlike for the case of ϵ' studied in the previous section, we found that the bound on T' is only very weakly correlated with the value of Δ_ϵ , and so we do not show separately the allowed regions in the $(\Delta_\sigma, \Delta_\epsilon)$ plane.

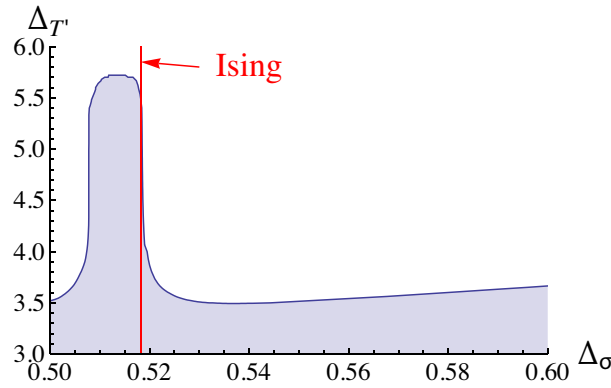


Figure 5.7: Upper bound on the dimension of the second spin 2 operator $T'_{\mu\nu}$ from the crossing symmetry constraint (5.24). The algorithm from Appendix 5.D was used with $n_{\max} = 10$. The 3D Ising vertical red line is five times wider than the error band in Table 5.1.

The $\Delta_{T'}$ bound shows fascinating non-monotonic behavior, generically hovering at around $\Delta_{T'} \sim 3.5$, but jumping to much higher allowed dimensions for a narrow range of Δ_σ . To begin

with, this implies that any moderate gap in the T' dimension, e.g. $\Delta_{T'} \geq 4$, leads to a sharp upper (as well as lower) bound on Δ_σ . Taken together with the plots in Fig. 5.5, one then obtains very small *closed* regions in the $(\Delta_\sigma, \Delta_\varepsilon)$ plane.

Furthermore, the narrow range of Δ_σ where large $\Delta_{T'}$ are allowed includes the 3D Ising value at its right end. The actual bound there is:

$$\Delta_\sigma \approx 0.518 \quad \implies \quad \Delta_{T'} \lesssim 5.5. \quad (5.25)$$

Unfortunately, Table 5.1 is mute about $\Delta_{T'}$ as we are not aware of any prior studies. However, we can get a rough estimate of this dimension by interpolating between 2D and 4D. In the 4D free scalar theory the first \mathbb{Z}_2 -even spin 2 operator after the stress tensor is

$$T'_{\mu\nu} = :\phi^2 T_{\mu\nu}: \quad (4D), \quad (5.26)$$

which has dimension 6. To be more precise, in the free scalar theory this operator is decoupled from the $\phi \times \phi$ OPE, but we expect it to couple in the Wilson-Fischer fixed point in $4 - \epsilon$ dimensions.

In the 2D Ising model the first such operator is

$$T' = L_{-4} \bar{L}_{-2} \mathbf{1} \quad (2D), \quad (5.27)$$

again of dimension 6. Notice that another 2D candidate, $L_{-2}\varepsilon$ of dimension 3, is not a quasi-primary but proportional to the $SL(2, \mathbb{C})$ descendant $L_{-1}^2 \varepsilon \equiv \partial_z^2 \varepsilon$ since the field $\varepsilon = \phi_{2,1}$ is degenerate on level 2 in the 2D Ising model. Still another candidate, $L_{-3} \bar{L}_{-1} \varepsilon$ of dimension 5, is an $SL(2, \mathbb{C})$ descendant of the spin 3 quasi-primary $L_{-3} \varepsilon$.

Assuming as usual that the 2D Ising and the 4D free scalar are continuously connected by the line of Wilson-Fischer fixed points to which the 3D Ising model also belongs, we expect by interpolation that $\Delta_{T'} \approx 6$ in 3D, not far from the upper end of the range allowed by the rigorous bound (5.25).

5.5.4 Bounds on Higher Spin Primaries

In addition to bounding operators in the scalar and spin 2 sectors, we can also attempt to place bounds on higher spin primaries in the $\sigma \times \sigma$ OPE. The first such operator in the 3D Ising model is the spin 4 operator $C_{\mu\nu\kappa\lambda}$. This operator is interesting because it controls the leading effects of rotational symmetry breaking when the 3D Ising model is placed on a cubic lattice. The corresponding perturbation of the CFT Lagrangian can be written as

$$\delta\mathcal{L}_{\text{CFT}} \propto C_{1111} + C_{2222} + C_{3333} . \quad (5.28)$$

Because of this connection with phenomenology, the dimension of C has been computed rather precisely: $\Delta_C \simeq 5.0208(12)$.

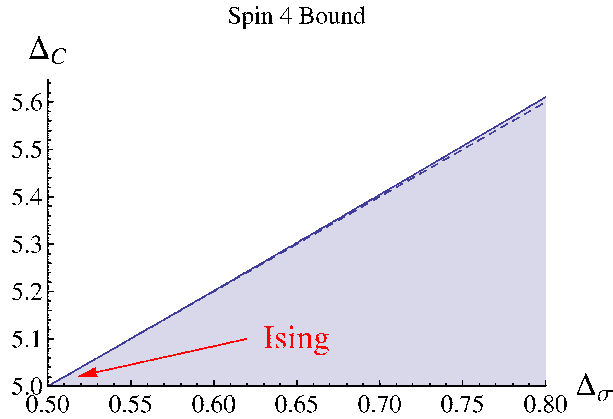


Figure 5.8: Upper bound on the dimension of the first spin 4 operator in the $\sigma \times \sigma$ OPE from the crossing symmetry constraint (5.24). The algorithm from Appendix 5.D was used with $n_{\text{max}} = 10$. The tip of the arrow shows the point $(\Delta_\sigma, \Delta_C)$ with the 3D Ising model values from Table 5.1. The dashed line is the gaussian solution $\Delta_4 = 2\Delta_\sigma + 4$.

In Fig. 5.8 we give a rigorous upper bound on Δ_C following from crossing symmetry and unitarity, making no other assumptions about the spectrum. While this bound passes above the value of Δ_C in the 3D Ising model, this is easily understood by the fact that the gaussian solution to crossing symmetry has $\Delta_C = 2\Delta_\sigma + 4$, which must be respected by our bound. The interesting and highly nontrivial statement is then that the gaussian solution seems to essentially saturate the

bound. The bound that we find is fit well by the curve:

$$\Delta_C^{\max} \simeq (2\Delta_\sigma + 4) + 0.0019(\Delta_\sigma - 1/2) + 0.1098(\Delta_\sigma - 1/2)^2 + O((\Delta_\sigma - 1/2)^3), \quad (5.29)$$

so that one can see that linear deviations are extremely suppressed, and quadratic deviations are at least somewhat suppressed. It is tempting to conjecture that the optimal bound (taking $n_{\max} \rightarrow \infty$) will exactly follow the gaussian line. It will be important in future studies to closely examine behavior of the bound at even larger external dimensions, to better understand whether deviations from this conjectured behavior are allowed.

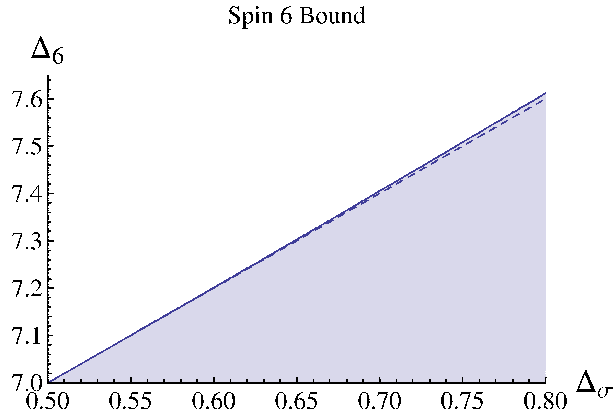


Figure 5.9: Upper bound on the dimension of the first spin 6 operator in the $\sigma \times \sigma$ OPE from the crossing symmetry constraint (5.24). The algorithm from Appendix 5.D was used with $n_{\max} = 10$. The dashed line is the gaussian solution $\Delta_6 = 2\Delta_\sigma + 6$.

Does this behavior of closely following the gaussian line hold for higher spins? To explore this, in Fig. 5.9 we show the analogous upper bound on the lowest-dimension spin 6 operator in the $\sigma \times \sigma$ OPE. This operator would control breaking of rotational symmetry on the tetrahedral lattice, but we are not aware of prior 3D studies of its dimension. Again we see that the bound closely follows the gaussian line $\Delta_6 = 2\Delta_\sigma + 6$, with a fit:

$$\Delta_6^{\max} \simeq (2\Delta_\sigma + 6) - 0.0020(\Delta_\sigma - 1/2) + 0.1388(\Delta_\sigma - 1/2)^2 + O((\Delta_\sigma - 1/2)^3), \quad (5.30)$$

so that again both linear and quadratic deviations are suppressed. We have verified that this trend

continues for operators of spin 8 and 10.

An important feature of these bounds is that they approach the dimensions of spin l conserved currents $\Delta_l = l + 1$ as $\Delta_\sigma \rightarrow 1/2$. It is well known that theories of free scalars contain higher spin conserved currents. Our bound shows that theories containing almost-free scalars necessarily contain higher spin operators that are almost conserved currents. A CFT version of the Coleman-Mandula theorem proved recently in [142] shows that theories containing higher spin currents and a finite central charge necessarily have the correlation functions of free field operators. This implies that we should also be able to derive a *lower* bound on the dimensions of higher spin operators, perhaps under the assumption of a finite central charge. It would be also interesting to connect these studies with an old result of Nachtmann [143] that in a unitary theory the *leading twists*

$$\tau_l = \Delta_l - (l + D - 2), \quad (5.31)$$

where Δ_l is the dimension of the lowest spin l operator, must form a *nondecreasing* and *convex upward* sequence for $l \geq 2$. We leave exploration of these very interesting directions to future work.

These bounds are also particularly interesting in the context of the AdS/CFT correspondence, since they place tight constraints on $O(1/N^2)$ corrections to the dimensions of double-trace operators. Concretely, free scalars in AdS give rise to spin- l double-trace operators with gaussian dimensions $2\Delta_\sigma + 2n + l$ for integer n , while bulk interactions generate $O(1/N^2)$ corrections to these dimensions. Some explicit examples of these corrections were studied, e.g., in [83, 84]. If our conjecture that the gaussian solution saturates the bound is true, then the bounds forbid bulk interactions that generate positive corrections to these dimensions, which in turn may imply positivity constraints on (higher derivative) interactions in AdS. Such constraints could then be related to the constraints on higher derivative interactions studied in [144]. This is clearly another direction worth studying in future work.

Finally, let us mention that similar bounds can be derived on the lowest dimension spin

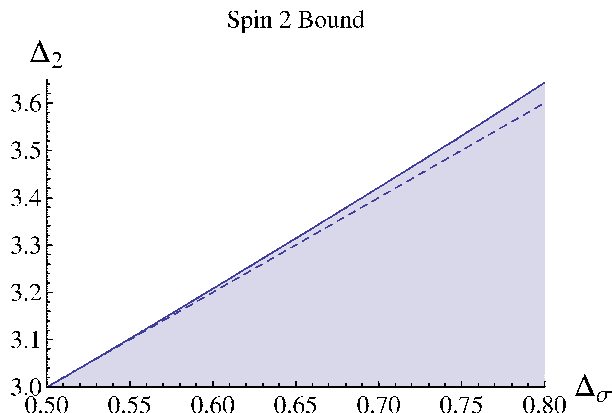


Figure 5.10: Upper bound on the dimension of the first spin 2 operator in the $\sigma \times \sigma$ OPE from the crossing symmetry constraint (5.24) in non-local theories without a stress tensor. The algorithm from Appendix 5.D was used with $n_{\max} = 10$. The dashed line is the gaussian solution $\Delta_2 = 2\Delta_\sigma + 2$.

2 operator in (non-local) theories where a stress tensor does not appear in the $\sigma \times \sigma$ OPE. This bound (Fig. 5.10) shows similar features to the higher spin bounds.

Such non-local theories may be interesting for several reasons. First, they commonly arise in statistical mechanics as models of long-range critical behavior. One much studied example is the critical point of the *long-range Ising model*, defined by a lattice Hamiltonian with a power-law spin-spin interaction:

$$\mathcal{H} = - \sum_{i,j} \frac{s_i s_j}{r_{ij}^{D+\gamma}}. \quad (5.32)$$

The precise universality class of this model depends on the value of γ . According to classic results [145, 146] supported by Monte-Carlo simulations [147], there are three regions. For γ sufficiently small, namely $\gamma \leq D/2$, the critical point is the gaussian model with the spin-field dimension determined by the naïve continuous limit of (5.32): $\Delta_\sigma = (D - \gamma)/2$. Then there is an intermediate region, and finally the region of large γ , in which the model belongs to the usual, short-range, Ising model universality class and the critical exponents do not depend on γ . The boundary between the intermediate and short-range region lies at $\gamma = D - 2\Delta_\sigma^{\text{Ising}}$, determined by the short-range Ising model spin-field dimension. This can be also understood by studying stability of the short-

range Ising model with respect to non-local perturbations. Analogously, the boundary between the gaussian and the intermediate region lies at the value of γ for which the operator σ^4 becomes marginal.

In the intermediate region, the σ dimension is still given by the mean-field formula $\Delta_\sigma = (D - \gamma)/2$, but the dimensions of other operators, such as ε , have nontrivial dependences on γ deviating from the gaussian values. So these fixed points are interacting. Because of their origin as relevant perturbations of the non-local gaussian scalar theory, they are expected to have conformal symmetry (and not just scale invariance), but not a stress-tensor. It is for such non-local CFTs that our bound in Fig. 5.10 may be of interest.

Another reason to be interested in theories without a stress tensor is that they realize a simpler case of AdS/CFT, in which bulk gravity is decoupled, so that the AdS metric is viewed as a fixed non-fluctuating background.⁶ This may be useful when one is interested in aspects of the correspondence which are not necessarily related to gravity, as e.g. in [83]. Also, removing gravity allows one to find nontrivial UV-complete AdS/CFT examples which are purely field-theoretic (no strings): any UV-complete quantum field theory on the AdS_{D+1} background can be interpreted as providing a dual description to a non-local D -dimensional CFT on the boundary.

5.5.5 Bounds on the Central Charge

Our final application concerns the central charge C_T of the 3D Ising model, defined for an arbitrary D as the coefficient of the canonically normalized stress tensor two-point function:

$$\begin{aligned} \langle T_{\mu\nu}(x) T_{\lambda\sigma}(0) \rangle &= \frac{C_T}{S_D^2} \frac{1}{(x^2)^D} \left[\frac{1}{2} (I_{\mu\lambda} I_{\nu\sigma} + I_{\mu\sigma} I_{\nu\lambda}) - \frac{1}{D} \delta_{\mu\nu} \delta_{\lambda\sigma} \right], \\ I_{\mu\nu} &= \delta_{\mu\nu} - 2x_\mu x_\nu / x^2, \quad S_D = 2\pi^{D/2} / \Gamma(D/2). \end{aligned} \quad (5.33)$$

⁶Such theories may alternately be viewed as the starting point for “constructive holography” by defining a CFT perturbatively around the $c \rightarrow \infty$ point as done in e.g. [148].

It seems that the 3D Ising central charge has been computed only to the second order in the ϵ -expansion, with the result [131, 149–151]

$$C_T/C_T^{\text{free}} = 1 - \frac{5}{324}\epsilon^2 + O(\epsilon^3), \quad (5.34)$$

where $C_T^{\text{free}} = D/(D-1)$ is the free scalar field central charge. Substituting $\epsilon \rightarrow 1$ and neglecting the unknown higher-order terms, this estimate would suggest that C_T/C_T^{free} is very close to 1, around 0.98 or so.

In our method, we can get control over C_T because the stress tensor conformal block enters the crossing symmetry constraint (5.24) with a C_T dependent coefficient:⁷

$$p_{D,2} = \frac{D}{D-1} \frac{\Delta_\sigma^2}{C_T}. \quad (5.35)$$

Following [57, 58], the conformal bootstrap can be used to bound the coefficient $p_{3,2}$ from above, which bounds the central charge from below. In Fig. 5.11 we show the lower bound on C_T as a function of Δ_σ . We see that the bound has a distinctive minimum close to the 3D Ising value of the σ dimension. The position of the minimum corresponds to $C_T/C_T^{\text{free}} \approx 0.94$. One may also redo the plot in Fig. 5.11 making some assumption about Δ_ϵ , like that $\Delta_\epsilon \geq \Delta_\sigma$. The most aggressive assumption would be to fix Δ_ϵ to the maximal value allowed by the upper bound in Fig. 5.3. One finds that the shape of the bound on C_T is very weakly dependent on these assumptions, but that the minimum moves to the right, even closer to the 3D Ising Δ_σ , and slightly higher up to $C_T/C_T^{\text{free}} \approx 0.95$.

We believe that the observed minimum in the C_T lower bound is not accidental, but must be close to the true value of C_T .⁸ This would imply a small but noticeable discrepancy with the ϵ -expansion estimate of C_T , which can be attributed to the unknown higher-order terms. In fact,

⁷The prefactor is different from [58] due to the different conformal block normalization, see Eq. (5.37).

⁸In 2D, a similar analysis reproduces the exact value of the 2D Ising model central charge with 10^{-4} accuracy [152].

we can also derive *upper* bounds on C_T in presence of a gap between T and T' . The strength of these bounds depends on the assumption about the gap, and for T' close to the maximal value allowed by (5.25) would rigorously rule out the ϵ -expansion estimate. We leave full exploration of such upper bound bounds to future work.

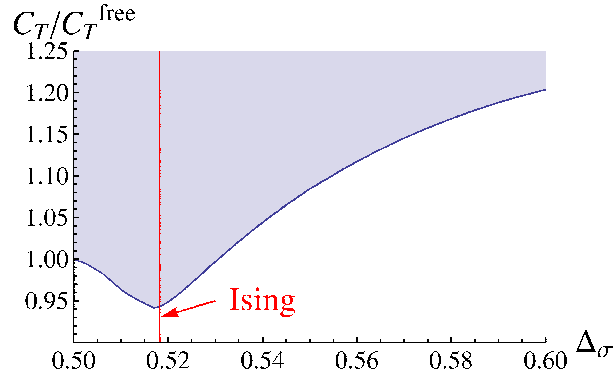


Figure 5.11: The lower bound on C_T as a function of Δ_σ . The plot was obtained with $n_{\max} = 11$. The 3D Ising vertical red line is five times wider than the error band in Table 5.1.

5.6 Discussion

The results of the previous section have many implications whose importance is hard to overestimate. First, all of our bounds are consistent with everything that was previously known about the critical exponents of the 3D Ising model, as computed via RG methods and measured in experiments and Monte Carlo simulations. We should take this a very strong evidence that the 3D Ising model has a full conformal symmetry, justifying post factum the use of conformal symmetry in studying this theory. It would be good to further test the conformal invariance experimentally or on the lattice, for example by measuring the form of the 3-point functions. One can also compare any new measurements (e.g., of the central charge) against the constraints obtained using the methods in this paper.

It is worth emphasizing that the bootstrap approach to studying 3D CFTs taken in this

paper has, in principle, a significant advantage over other methods – at every step in the program we can present constraints that are completely rigorous (up to numerical errors that can be made arbitrarily small). This is significantly better than the usual situation in field theory computations, where one computes the first several terms in a series (say, the ϵ -expansion or a loop expansion) and one can only estimate the errors from neglecting higher terms. It is also an advantage over lattice simulations, where it can be very difficult to gain control over errors induced by discretizing the theory.

In this paper we have imposed only the first and the simplest of the infinitely many bootstrap conditions – the one following from the crossing symmetry of the σ four-point function. It turns out that this condition alone carves out a significant portion of the operator dimension space. The 3D Ising model seems to lie on the boundary of the allowed region, and at a rather special point – a corner. This empirical fact suggests that the model is algebraically special, for two reasons. First, the crossing symmetry constraint is expected to allow fewer solutions at the boundary of the allowed region as compared to the bulk, perhaps just a unique solution. Second, the non-analytic behavior of the bound at a corner point can be attributed to rapid rearrangements of the operator spectrum [62]. Indeed, Figs. 5.6 and 5.7 show rapid changes happening for the next-to-leading operator dimensions in the scalar and spin 2 sectors. Such spectrum rearrangements signal linear (near-)degeneracies among various conformal blocks. It is very important to explore this phenomenon in detail as it offers tantalizing hope for distilling some *analytical* understanding of the 3D Ising model dimensions from our numerical approach. More generally, the fact that some special theories seem to lie at the edge of the region allowed by crossing symmetry may suggest a new classification scheme for understanding CFTs in $D > 2$.

Furthermore, it is intriguing that most of our bounds (not just Δ_ϵ) seem to be essentially saturated by the values realized in the 3D Ising model. This fact suggests the strategy of determining the spectrum recursively: first fix ϵ at the maximal allowed dimension, then ϵ' at the maximal

allowed dimension given Δ_ε , etc. We hope to explore the viability of this approach (perhaps also including gaps in higher spin operators) in future work.

Another badly needed development is to add conformal bootstrap constraints coming from other correlators, which can lead to interesting interplay. For example, we would like to include $\langle\sigma\varepsilon\sigma\varepsilon\rangle$ expanding in the $\sigma \times \varepsilon$ channel, since this expansion will be crossing symmetric. Moreover, the conformal block of σ will appear with the same coefficient $f_{\sigma\sigma\varepsilon}^2$ as the conformal block of ε in the analysis of $\langle\sigma\sigma\sigma\sigma\rangle$. It is also interesting to include $\langle\varepsilon\varepsilon\varepsilon\varepsilon\rangle$ whose expansion involves the same \mathbb{Z}_2 -even operators as $\langle\sigma\sigma\sigma\sigma\rangle$. Due to (5.35), the stress tensor will appear in both expansions with related coefficients. One can also consider 4-point functions containing the stress tensor, where the recent results of [68] on conformal blocks for external operators with spin can be used.

Another future task is to study 3D CFTs with larger global symmetry groups, such as $O(N)$ symmetry. The general theory of analyzing bootstrap constraints in the presence of a continuous global symmetry was given in [59]. The equations look more difficult as the OPE contributions should be classified into various representations and the crossing symmetry transformation involves a Fierz matrix. Nevertheless, there are always as many equations as representation channels and the total constraining power is expected to be comparable to the \mathbb{Z}_2 -symmetric case. In 4D, this has been convincingly demonstrated in [60, 61], where many strong bounds for $O(N)$ and $SU(N)$ symmetric CFTs have been obtained. It would be interesting to generalize these methods to 3D and see how the resulting bounds on operator dimensions compare to what is known about the $O(N)$ -vector models.

Cross-fertilizing in the opposite direction, it is worth applying in 4D what we have learned in this paper in the 3D context – how interesting it is to study the effects of gaps in the operator spectrum. In addition, we should stress that the recursion relations for conformal blocks exhibited in this paper are valid for any space-time dimension D . Thus, we can use them to numerically compute conformal blocks in $4 - \epsilon$ dimensions for different values of ϵ , where we can make contact

with operator dimensions and OPE coefficients computed perturbatively in the ϵ -expansion.

Our results and discussion in Section 5.5.4 show that one can also learn interesting statements about higher spin operators from crossing symmetry. It will be interesting to explore what can be learned further, particularly in the context of the AdS/CFT correspondence, where for example the $O(N)$ -vector models in the large N limit are described by higher spin gauge theories in AdS_4 [153, 154]. There is clearly still much to be learned about the role that higher spin operators play in ensuring consistency of the theory, and about how gaps in the lower-spin spectrum affect what these operators are allowed to do.

Overall, our results fly in the face of the prevailing opinion that above two dimensions conformal symmetry by itself is not sufficiently restrictive to solve models. Clearly, the conformal bootstrap in $D > 2$ works. We have not yet solved the 3D Ising model, but we have definitely cornered it.

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5.A Recursion Relations at Fixed External Dimensions

Our conformal blocks are the same as the functions F_{λ_1, λ_2} of Ref. [125]:

$$G_{\Delta, l} = F_{\lambda_1, \lambda_2}, \quad \lambda_1 = \frac{1}{2}(\Delta + l), \quad \lambda_2 = \frac{1}{2}(\Delta - l). \quad (5.36)$$

This normalization is different from the one used in a number of previous works. E.g., in [64, 68] conformal blocks contain an extra factor of⁹

$$\frac{(2\alpha)_l}{(-2)^l(\alpha)_l}, \quad \alpha \equiv \frac{D}{2} - 1. \quad (5.37)$$

This follows by comparing Eq. (2.25-29) of [125] with Eq. (2.22) of [68].

We note in passing one reason for using the new normalization: once conformal blocks are analytically continued to all real l , one has a symmetry relation ([125], Eq. (4.10))

$$G_{\Delta, l} = G_{\Delta, -l-D+2}. \quad (5.38)$$

In particular, we have $G_{\Delta, -1} = G_{\Delta, 0}$ in 3D, which can be useful as explained in footnote 3.

Below we consider only the case $\Delta_{12} = \Delta_{34} = 0$, which corresponds to setting $a = b = 0$ in the notation of [125]. Denote

$$\beta_p \equiv \frac{p^2}{4(2p-1)(2p+1)}, \quad D_z \equiv z^2(1-z)\partial_z^2 - z^2\partial_z, \quad (5.39)$$

$$\mathcal{F}_0 \equiv \frac{1}{z} + \frac{1}{\bar{z}} - 1, \quad \mathcal{F}_1 \equiv (1-z)\partial_z + (1-\bar{z})\partial_{\bar{z}}, \quad \mathcal{F}_2 \equiv \frac{z-\bar{z}}{z\bar{z}}(D_z - D_{\bar{z}}). \quad (5.40)$$

⁹ α was called ϵ in [125], but we already have two other epsilons in this paper.

It was shown in [125] that $\mathcal{F}_i F_{\lambda_1 \lambda_2}$ can be expressed as linear combinations of $F_{\lambda'_1 \lambda'_2}$. More specifically, we have (see [125], Eqs. (4.28), (4.29), (4.32))

$$\begin{aligned} \mathcal{F}_0 F_{\lambda_1 \lambda_2} &= \frac{l+2\alpha}{l+\alpha} F_{\lambda_1 \lambda_2-1} + \frac{l}{l+\alpha} F_{\lambda_1-1 \lambda_2} \\ &+ \frac{(\Delta-1)(\Delta-2\alpha)}{(\Delta-1-\alpha)(\Delta-\alpha)} \left(\frac{l+2\alpha}{l+\alpha} \beta_{\lambda_1} F_{\lambda_1+1 \lambda_2} + \frac{l}{l+\alpha} \beta_{\lambda_2-\alpha} F_{\lambda_1 \lambda_2+1} \right), \end{aligned} \quad (5.41)$$

$$\begin{aligned} \mathcal{F}_1 F_{\lambda_1 \lambda_2} &= \frac{l+2\alpha}{l+\alpha} \lambda_2 F_{\lambda_1 \lambda_2-1} + \frac{l}{l+\alpha} (\lambda_1 + \alpha) F_{\lambda_1-1 \lambda_2} + \frac{(\Delta-1)(\Delta-2\alpha)}{(\Delta-1-\alpha)(\Delta-\alpha)} \times \\ &\times \left(\frac{l+2\alpha}{l+\alpha} (-\lambda_1 + \alpha + 1) \beta_{\lambda_1} F_{\lambda_1+1 \lambda_2} + \frac{l}{l+\alpha} (-\lambda_2 + 2\alpha + 1) \beta_{\lambda_2-\alpha} F_{\lambda_1 \lambda_2+1} \right), \end{aligned} \quad (5.42)$$

$$\begin{aligned} \mathcal{F}_2 F_{\lambda_1 \lambda_2} &= (\Delta-1) \frac{l(l+2\alpha)}{l+\alpha} \left[F_{\lambda_1 \lambda_2-1} - F_{\lambda_1-1 \lambda_2} \right. \\ &\left. - \frac{(\Delta-2\alpha)(\Delta-1-2\alpha)}{(\Delta-1-\alpha)(\Delta-\alpha)} (\beta_{\lambda_1} F_{\lambda_1+1 \lambda_2} - \beta_{\lambda_2-\alpha} F_{\lambda_1 \lambda_2+1}) \right], \end{aligned} \quad (5.43)$$

where $\Delta = \lambda_1 + \lambda_2$, $l = \lambda_1 - \lambda_2$.

Let us now view (5.41) and (5.43) as a linear 2×2 system for the spin $l+1$ conformal blocks $F_{\lambda_1+1 \lambda_2}$ and $F_{\lambda_1 \lambda_2-1}$. Eliminating one of these, say $F_{\lambda_1+1 \lambda_2}$, we get a recursion relation expressing the remaining spin $l+1$ block in terms of spin l and spin $l-1$ blocks only. Shifting the spin by one and passing to the $G_{\Delta,l}$ notation, this relation takes the form:

$$\begin{aligned} \frac{(\Delta-\alpha)(l+2\alpha-1)}{l+\alpha-1} G_{\Delta,l} &= \frac{\alpha(\Delta+l-1)}{l+\alpha-1} G_{\Delta,l-2} + \frac{1}{2} \left((\Delta-2\alpha) \mathcal{F}_0 + \frac{\mathcal{F}_2}{l-1} \right) G_{\Delta+1,l-1} \\ &- \frac{\Delta(\Delta-2\alpha)(\Delta-2\alpha+1)}{(\Delta-\alpha)(\Delta-\alpha+1)} \beta_{\frac{1}{2}(\Delta-l+2-2\alpha)} G_{\Delta+2,l-2}. \end{aligned} \quad (5.44)$$

When we specialize to the line $z = \bar{z}$, the term involving \mathcal{F}_2 vanishes. We are then left with a nonderivative recursion relation, Eq. (5.13) of the main text.

Alternatively, we can apply the same logic to the system formed by (5.41) and (5.42). Eliminating again $F_{\lambda_1+1 \lambda_2}$ in favor of $F_{\lambda_1 \lambda_2-1}$, shifting the spin by one and passing to the $G_{\Delta,l}$

notation, we get:

$$\begin{aligned} \frac{(\Delta - \alpha)(l + 2\alpha - 1)}{(l + \alpha - 1)} G_{\Delta, l} &= \left(\frac{1}{2}(\Delta + l - 2\alpha - 2)\mathcal{F}_0 + \mathcal{F}_1 \right) G_{\Delta+1, l-1} \\ &- (l - 1) \left(\frac{\Delta(\Delta - 2\alpha + 1)}{(\Delta - \alpha)(\Delta - \alpha + 1)} \beta_{\frac{1}{2}(\Delta - l + 2 - 2\alpha)} G_{\Delta+2, l-2} + \frac{\Delta + l - 1}{l + \alpha - 1} G_{\Delta, l-2} \right). \end{aligned} \quad (5.45)$$

Recursions (5.44) and (5.45) have complementary advantages. The first one becomes nonderivative at $z = \bar{z}$ and can be used to compute high spin blocks on this line efficiently. However, it needs both $l = 0$ and $l = 1$ blocks to start up (except in $D = 3$ where it can be started from $l = 0$ and $l = -1$, but we would like a framework which works in any D). On the other hand, recursion (5.45) has spin $l - 2$ blocks entering with a factor $(l - 1)$ and can be started up with just spin 0. In Appendix 5.B, we'll use (5.45) to compute spin 1 blocks at $z = \bar{z}$ from spin 0, but switch to the nonderivative recursion (5.44) for higher spins.

5.B Scalar and Spin 1 Blocks at $z = \bar{z}$

In this appendix we'll derive formulas for the spin 0 and 1 conformal blocks at $z = \bar{z}$ for equal external dimensions.

We start with the double series expansion (5.11) for the scalar conformal block. Performing the summation in n , we get ($\alpha = D/2 - 1$)

$$G_{\Delta, 0} = \sum_{m=0}^{\infty} \frac{\left(\left(\frac{\Delta}{2}\right)_m\right)^4}{m!(\Delta)_{2m}(\Delta - \alpha)_m} u^{\frac{\Delta}{2} + m} {}_2F_1\left(m + \frac{\Delta}{2}, m + \frac{\Delta}{2}; 2m + \Delta; 1 - v\right). \quad (5.46)$$

We now replace ${}_2F_1$ by its Euler integral representation

$${}_2F_1(a, b; c; x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 dt \frac{t^{b-1}(1-t)^{c-b-1}}{(1-tx)^a}. \quad (5.47)$$

The series in m under the integral sign turns out to be hypergeometric in the variable

$$X = \frac{(1-t)tu}{1-t(1-v)}, \quad (5.48)$$

so that we find:

$$G_{\Delta,0} = \frac{\Gamma(\Delta)}{\Gamma(\frac{\Delta}{2})^2} \int_0^1 \frac{dt}{t(1-t)} X^{\Delta/2} {}_2F_1\left(\frac{\Delta}{2}, \frac{\Delta}{2}; \Delta - \alpha; X\right). \quad (5.49)$$

Now let us use the hypergeometric identity

$${}_2F_1(a, b; c; x) = (1-x)^{-b} {}_2F_1\left(c-a, b; c; \frac{x}{x-1}\right). \quad (5.50)$$

The resulting expression factorizes nicely in terms of z and \bar{z} :

$$G_{\Delta,0} = \frac{\Gamma(\Delta)}{\Gamma(\frac{\Delta}{2})^2} \int_0^1 \frac{dt}{t(1-t)} Y^{\Delta/2} {}_2F_1\left(\frac{\Delta}{2}, \frac{\Delta}{2} - \alpha; \Delta - \alpha; -Y\right), \quad (5.51)$$

$$Y \equiv \frac{X}{1-X} = \frac{t(1-t)z\bar{z}}{(1-tz)(1-t\bar{z})}. \quad (5.52)$$

Now replace ${}_2F_1$ by its defining power series expansion in $(-Y)$ and integrate the series term by term. For $z = \bar{z}$, the resulting integrals are of the form (5.11) and give hypergeometric functions ${}_2F_1(\Delta + 2n, \Delta/2 + n; \Delta + 2n; z)$, which are elementary.¹⁰ So we get:

$$G_{\Delta,0}|_{z=\bar{z}} = \left(\frac{z^2}{1-z}\right)^{\Delta/2} \sum_{n=0}^{\infty} \frac{[(\Delta/2)_n]^3 (\Delta/2 - \alpha)_n}{n! (\Delta)_{2n} (\Delta - \alpha)_n} \left(\frac{z^2}{z-1}\right)^n. \quad (5.53)$$

Expressing $(\Delta)_{2n}$ via the duplication formula for the Γ function, the series is recognized to be of the ${}_3F_2$ type, and we get precisely Eq. (5.14).

Is there a similar closed form representation for generic unequal external dimensions or, more specifically, for generic $\Delta_{12} = \Delta_{34} \neq 0$ (as would be needed for the crossing symmetry analysis of the $\langle \sigma \varepsilon \sigma \varepsilon \rangle$ correlator)? The following reasoning shows that this may be difficult. For $D = 2$, Eq. (5.14) can be derived starting from the explicit expression (5.8), passing to the variable $z^2/(4(z-1))$ via the identity

$${}_2F_1(a, b, 2b, z) = \left(1 - \frac{z}{2}\right) (1-z)^{-\frac{a+1}{2}} {}_2F_1\left(\frac{1-a+2b}{2}, \frac{a+1}{2}, b + \frac{1}{2}; \frac{z^2}{4(z-1)}\right), \quad (5.54)$$

and then aiming for Clausen's formula ([155], Sec. 4.3) to express the square of a ${}_2F_1$ as a ${}_3F_2$.

However, Eq. (5.54) is not useful for generic unequal dimensions.

¹⁰For $z \neq \bar{z}$ we would have obtained a series in Appel F_1 functions.

Passing to the spin 1 case, the idea is to use the second recursion relation (5.45) which expresses spin 1 blocks via the spin 0 ones. This relation can be restricted to the $z = \bar{z}$ line, as the differential operator \mathcal{F}_1 acts within the line; for $l = 1$ it gives

$$G_{\Delta,1}(z) = \frac{1}{2(\Delta - \alpha)} \left[\frac{2-z}{2z}(\Delta - 2\alpha - 1) + (1-z)\partial_z \right] G_{\Delta+1,0}(z). \quad (5.55)$$

Substituting the spin 0 closed form expression (5.14), we find

$$G_{\Delta,1}(z) = \frac{2-z}{2(\Delta - \alpha)z} \left(\frac{z^2}{1-z} \right)^{\frac{\Delta+1}{2}} [y \partial_y + \Delta - \alpha] f(y), \quad (5.56)$$

where $f(y)$ (with $y \equiv z^2/4(z-1)$) is the ${}_3F_2$ function entering the expression for $G_{\Delta+1,0}(z)$. Eq. (5.15) then follows, since the ${}_3F_2$ function satisfies

$$[y \partial_y + b_2 - 1] {}_3F_2(a_1, a_2, a_3; b_1, b_2; y) = (b_2 - 1) {}_3F_2(a_1, a_2, a_3; b_1, b_2 - 1; y). \quad (5.57)$$

5.C Recursion Relation for the Transverse Derivatives

The following recursion relation for $h_{m,n}$ can be derived by applying $\partial_a^m \partial_b^n$ to the Casimir equation (5.17) written in the a, b coordinates, setting $a \rightarrow 1$, $b \rightarrow 0$, and shifting $n \rightarrow n - 1$:

$$\begin{aligned} 2(D + 2n - 3)h_{m,n} = & \\ & 2m(D + 2n - 3)[-h_{m-1,n} + (m-1)h_{m-2,n} + (m-1)(m-2)h_{m-3,n}] \\ & - h_{m+2,n-1} + (D - m - 4n + 4)h_{m+1,n-1} \\ & + [2C_{\Delta,l} + 2D(m+n-1) + m^2 + 8mn - 9m + 4n^2 - 6n + 2] h_{m,n-1} \\ & + m[D(m-2n+1) + m^2 + 12mn - 15m + 12n^2 - 30n + 20] h_{m-1,n-1} \\ & + (n-1)[h_{m+2,n-2} - (D - 3m - 4n + 4)h_{m+1,n-2}]. \end{aligned} \quad (5.58)$$

5.D Linear Programming Implementation

Let us write the crossing constraint Eq. (5.24) as

$$0 = F_{0,0}^{\Delta_s}(u, v) + \sum' p_{\Delta,l} F_{\Delta,l}^{\Delta_\sigma}(u, v), \quad (5.59)$$

where $F_{\Delta,l}^{\Delta_\sigma}(u, v) \equiv v^{\Delta_\sigma} G_{\Delta,l}(u, v) - u^{\Delta_\sigma} G_{\Delta,l}(v, u)$. To rule out some spectrum of operator dimensions, it suffices to find a linear functional Λ acting on functions of (u, v) such that

1. $\Lambda(F_{0,0}^{\Delta_\sigma}) = 1$ (normalization condition)
2. $\Lambda(F_{\Delta,l}^{\Delta_\sigma}) \geq 0$ for all Δ, l in the spectrum (positivity constraints).

Any such Λ would be inconsistent with the crossing relation Eq. (5.59) and positivity of the coefficients $p_{\Delta,l}$, implying that the putative spectrum cannot be realized in a unitary (or reflection positive) CFT.

In practice, we consider Λ of the form

$$\Lambda : F(u, v) \mapsto \sum_{m+2n \leq 2n_{\max}+1} \lambda_{m,n} \partial_a^m \partial_b^n F(a, b)|_{a=1, b=0} \quad (5.60)$$

where the variables a, b are defined in Eq. (5.19), $\lambda_{m,n}$ are real coefficients, and the range of m, n depends on an integer n_{\max} . Since $F_{\Delta,l}^{\Delta_\sigma}(u, v)$ is antisymmetric under $u \leftrightarrow v$, only odd a -derivatives are nonzero, and a given n_{\max} corresponds to $(n_{\max} + 1)(n_{\max} + 2)/2$ nonzero coefficients $\lambda_{m,n}$. Larger n_{\max} gives stronger bounds, but is more computationally intensive. Derivatives of $F_{\Delta,l}^{\Delta_\sigma}$ are simply linear combinations of derivatives of the conformal blocks $G_{\Delta,l}$, which we compute in **Mathematica** using the methods outlined in Section 5.4. We first evaluate the derivatives $\partial_a^m G_{\Delta,l}$ up to $m = 2n_{\max} + 1$ and all the other derivatives in the range $m + 2n \leq 2n_{\max} + 1$ follow via the recursion relation of Appendix 5.C. To compare with previous work, we have $n_{\max} = N/2 = k - 1$ where N and k are the parameters used in [55] and [61], respectively.

We implement the positivity constraints above by discretizing the set of dimensions and restricting the spin to lie below some large finite value. The conformal blocks $G_{\Delta,l}(u, v)$ converge

quickly for large dimensions and spins, so this is a reasonable approximation. It can be made arbitrarily good by using finer discretizations and a larger maximum dimension and spin. The plots in this paper were generated with the choices given in Table 5.2.

	δ	Δ_{\max}	L_{\max}
T1	2×10^{-5}	3	0
T2	5×10^{-4}	8	6
T3	0.02	100	50
T4	1	500	100

Table 5.2: In this work, we used a combination of four tables T1-T4 of conformal blocks (and their derivatives) with different discretizations, maximum dimensions, and maximum spins. For each table, dimensions were chosen from the unitarity bound $\Delta_{\min} \equiv l + 1 - \frac{1}{2}\delta_{l,0}$ to $\Delta_{\max} + 2(L_{\max} - l)$ with step δ , and spins were restricted to $0 \leq \lambda \leq L_{\max}$. The choices above allow for high-resolution studies of the low-spin spectrum (T1,T2), while simultaneously ensuring control of intermediate dimensions and spins (T3), and also asymptotic behavior (T4).

After restricting the dimensions and spins to lie in a finite set, our problem becomes a standard linear programming problem which can be solved on a computer. Solvers are available in a wide variety of software libraries and applications, including for example **Mathematica**. Here, we choose to use the dual simplex algorithm implementation in IBM’s **ILOG CPLEX Optimizer**.¹¹

To generate plots like those in Figure 5.5, we must scan over different choices of dimensions $\Delta_\sigma, \Delta_\epsilon, \dots$, solving a linear program each time to determine the boundary between feasible and infeasible choices. When scanning over a single dimension, for example, this is most efficiently done using a binary search. One can additionally generalize binary searches to work in higher dimensions by recursively refining a lattice of points. These algorithms are readily parallelizable, and it is very convenient to take advantage of a cluster of machines to perform the computations. Our search logic is implemented in **Scala**, taking advantage of its actor model for distributing parallel tasks across a network, and **ILOG CPLEX’s Java (Scala compatible) API** for performing the computations.

¹¹<http://www-01.ibm.com/software/integration/optimization/cplex-optimizer/>

Appendix A

A Helpful Method for Writing Physics Papers

```
#####  
# snarXiv.ml  
  
type phrase = Str of string | Opts of phrase array array  
  
let _ = Random.self_init ()  
  
let randelt a = a.(Random.int (Array.length a))  
let rec print_phr = match_phr with  
  Str s      -> print_string s  
| Opts options ->  
  let parts = randelt options in  
  Array.iter print parts  
  
let _ = print_top  
  
#####  
# snarXiv.grammar  
  
top ::= <paper>  
  
# by David Simmons-Duffin (http://www.physics.harvard.edu/~davidsd)  
# March 2010  
#  
# This grammar is free from context, and also free for you to use  
# however you like, although it's probably not a good idea to try  
# actually submitting any of these to the arXiv. Feel free to suggest
```

```

# improvements or additions, particularly famous physicists or physics
# concepts with funny names that I forgot.
#
# The code grew organically over several hours, so it may be poorly
# organized, incomplete, and inconsistent. Hopefully the output
# reflects that.

##### Numbers #####

zdigit ::= 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9
nzdigit ::= 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9
smallinteger ::= <nzdigit> | <nzdigit><zdigit> | <nzdigit><zdigit>
n ::= n | m | <nzdigit>

##### Basic Algebra #####

ring ::= \Z | \Q | \R | \C | \mathbb{H}
group ::= <liegroup> | <discretegroup>
liegroup ::= SU(<n>) | Sp(<n>) | SO(<n>) | G_2 | F_4 | E_6 | E_7 | E_8 |
            Spin(<n>)
discretegroup ::= \Z | \Z_<n> | \Z^<n> | Hom(<ring>,<ring>) | H^<n>(<mathspace>,<ring>)
                | H_<n>(<mathspace>,<ring>) | Ext^<n>(<ring>,<ring>) | M_<n>(<ring>)
                | SL_<n>(<ring>) | Dih_<n>
groupaction ::= orbifold | quotient

##### Spaces #####

space ::= <pluralspace> | <singspace> | <mathspace>
singspace ::= a <spacetype> | a <spaceadj> <spacetype> | <prospacename>
            | <spaceadj> <prospacename> | <mathspace> | <mathspace>
            | a <bundletype> bundle over <space> | <singspace> fibered over <singspace>
            | the moduli space of <pluralspace> | a <spacetype> <spaceproperty>
            | the <spacepart> of <space> | a <group> <groupaction> of <singspace>
            | the near horizon geometry of <singspace>
pluralspace ::= <spacetype>s | <spaceadj> <spacetype>s | <n> copies of <mathspace> |
               <pluralspace> fibered over <space> | <spacetype>s <spaceproperty> |
               <bundletype> bundles over <space> | moduli spaces of <pluralspace> |
               <group> <groupaction>s of <pluralspace>
spaceadj ::= <spaceadj> <spaceadj> | warped | squashed | non-compact | compact |
            hyper-Kahler | Kahler | exotic | projective | noncommutative | fuzzy |
            elliptically-fibered | spin | hyperbolic | Einstein | Ricci-flat |
            Euclidean | Minkowskian | planar | harmonic | symplectic | ALE | ALF
spaceproperty ::= of <group> holonomy | with <mathadj> <mathobj>
bundletype ::= <group> | line | affine | <mathspace>
spacetype ::= Calabi-Yau <n>-fold | Riemann surface | manifold | <n>-fold | <n>-manifold |
            symmetric space | K3 | del-Pezzo | Klebanov-Strassler background |
            RS1 background | lens space | Hirzebruch surface | Enriques surface |
            rational surface | ALE fibration | ALF space | linear dilaton background |
            Atiyah-Hitchin manifold
generalspacetype ::= surface | line | hyperplane | hypersurface

```

```

properspacename ::= Anti de Sitter Space | de Sitter Space | Taub-NUT Space | superspace
mathspace ::= AdS_<n> | S^<n> | R^<n> | CY_<n> | C^<n>
              | dS_<n> | T^<n> | <mathspace> x <mathspace> | P^<n>
spacepart ::= boundary | conformal boundary | null future | horizon | NUT

##### More Mathematics #####

mapping ::=
    function | mapping | homomorphism | homeomorphism | isomorphism
    | surjective <mapping> | injective <mapping> | holomorphism
    | biholomorphism | isometry
mathadj ::= trivial | nontrivial | zero | nonzero | general | discrete | abelian
           | non-abelian | equivariant | <symmetry> symmetric
mathobj ::= fundamental group | cohomology | homology | torsion | monodromy
           | spin-structure | dimension | complex-structure | flux | B-field
           | H-flux
representation ::= adjoint | symmetric tensor | antisymmetric tensor | singlet |
                  doublet | triplet

##### Theories #####

theory ::= <singtheory> | <pluraltheory>
singtheory ::= <singqft> <theorymodifier> | <singstringtheory> <theorymodifier>
pluraltheory ::= <pluralqft> <theorymodifier> | <pluralstringtheory> <theorymodifier>
theorymodifier ::= || <compactified> on <space> | deformed by <operator>s | on <space>
                  | <near> <theoryobj>
compactified ::= living | compactified | dimensionally reduced | supported
near ::= in the presence of | near | surrounded by | far from
qft ::= <singqft> | <singqft> | <singqft> | <pluralqft>
singqft ::= <properqft> | <qftadj> <properqft> | <properqft> <qftproperty>
           | a <qftadj> <genericqft>
pluralqft ::= <qftadj> <genericqft>s
qftadj ::= | <qftadj> <qftadj> | supersymmetric | N=<nzdigit> | adjoint
           | superconformal | conformal | extremal | chiral | topological
           | <n>-dimensional | twisted | WZW | topologically twisted | deformed
           | perturbative | nonperturbative | Toda
qftproperty ::= with <qftobj>
qftobj ::= a <operator> | <operator>s | <mathadj> superpotential | a <optype> defect
           | <representation> <field>s | a <representation> <field>
           | gauge group <liegroup> | a <mathadj> deformation
           | <mathadj> kahler potential
genericqft ::= QFT | CFT | Matrix Model | TQFT | <theorytype> Theory
theorytype ::= Effective Field | Quantum Field | Conformal Field | String
              | Topological Field | Heavy Quark Effective | low-energy Effective
              | Yang-Mills | Chern-Simons | Soft-Collinear Effective | gauge
properqft ::= QCD | QED | supergravity | unparticle physics
stringtheory ::= <singstringtheory> | <pluralstringtheory>
singstringtheory ::= String Theory | F-Theory | M-Theory | Heterotic string theory
                  | Topological String Theory | type IIA | type IIB
pluralstringtheory ::= String theories | Heterotic strings | type IIA strings

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| type IIB strings | type I strings | topological strings
| bosonic strings

##### Physics Objects #####

theoryobj ::= <singtheoryobj> | <pluraltheoryobj>
singtheoryobj ::= a <bhadj> black hole | a <singularityadj> singularity
| a <branetype> brane <braneaction> | a <generalspacetype> defect
| a <branetype> instanton | an instanton | a <branetype> brane probe
| a stack of <branetype> branes <braneaction> | an orientifold plane
pluraltheoryobj ::= <bhadj> black holes | <singularityadj> singularities
| <branetype> branes <braneaction> | <generalspacetype> defects
| orientifold planes | <branetype> instantons | instantons
bhadj ::= orientifold | BTZ | Kerr | Reisner-Nordstrom | small | large
| Schwarzschild | <branetype> brane | massive | extremal
singularityadj ::= A_<n> | B_<n> | C_<n> | D_<n> | E_6 | E_7 | E_8 | G_2
| F_4 | conifold | conical | ADE | orbifold | du Val | Kleinian
| rational double-point | canonical | exceptional | <physicist>
branetype ::= NS5 | D<nzdigit> | (p,q) 7- | (p,q) | noncommutative | black
| fractional D<nzdigit> | special lagrangian | canonical co-isotropic
| holomorphic | A-type | B-type
braneaction ::= | wrapping a <mathspace> | wrapped on <singspace>
operator ::= <optype> operator | Chern-Simons term | <optype> F-term | Wilson line |
't Hooft line | <generalspacetype> operator | <optype> D-term
optype ::= primary | quasi-primary | marginal | relevant | irrelevant
| four-quark | multi-fermion | loop | local | nonlocal | BPS
field ::= boson | fermion | gauge-field | <n>-form | scalar
objectplace ::= at the center of the galaxy | in our solar system
| on the surface of the sun | at the Tevatron | at the GUT scale
| at the edge of our universe | in the CMB | at the LHC
| at SNO | at ATLAS | in the interstellar medium | at DAMA | at CDMS
| in the early universe | during inflation | after reheating
| at the weak scale | at  $\Lambda_{\text{QCD}}$  | at the intermediate scale
| at the Planck scale

##### Model #####

model ::= <singmodel> | <pluralmodel>
singmodel ::= a model of <physsubject> | a model for <physsubject>
| a <physadj> model <modelmodifier> | the <propermodel>
| the <physadj> <propermodel> | <physadj> <generalmodel>
| <inflationadj> inflation | <generalmodel> | <generalmodel>
| <physicist> <generalmodel>
pluralmodel ::= models of <physsubject> | <physadj> models <modelmodifier>
| models of <particle>s
modelmodifier ::= | of <physsubject> | for <physsubject> | with <particle>s
propermodel ::= Standard Model | MSSM | <nnnn>MSSM | Thirring Model | Ising Model
| XXZ Model | O(n) Model | <physicist> Model | Landau-Ginzburg Model
| A-model | B-model
nnnn ::= N | N<nnnn>

```

```

generalmodel ::= gravity | general relativity | RS1 | RS2 | technicolor
              | gauge mediation | anomaly mediation | <properqft>
              | <dynadjective> mechanics | <dynadjective> dynamics | hydrodynamics
              | thermodynamics | unparticle physics
dynadjective ::= quantum | <physicist> | <physadj>

##### Adjectives #####

physadj ::= <physadj> <physadj> | non-<physadj>
          | <nondescriptivephysadj> | <descriptivephysadj>
          | <nondescriptivephysadj> | <descriptivephysadj>
          | <nondescriptivephysadj> | <descriptivephysadj>
          | <nondescriptivephysadj> | <descriptivephysadj>
          | <nondescriptivephysadj> | <descriptivephysadj>
          | <nondescriptivephysadj> | <descriptivephysadj>
          | <nondescriptivephysadj> | <descriptivephysadj>
nondescriptivephysadj ::=
    seesaw | curvaton | hybrid | quantum | loop | cosmon
    | scalar | <particle> | <physsubject> | isocurvature | <branetype> brane
    | condensate | three-fluid | multi-field | variable mass
    | particle | matrix | lattice | inflaton | bulk | boundary | halo
    | braneworld | GUT | <liegroup> | scalar field | RS
    | flavor | Landau-Ginzburg | Planck | <physicist> | left-right
    | large-N | parent | QCD | QED | BPS | unparticle | high-scale | low-scale
    | large mass
descriptivephysadj ::=
    non-gaussian | simple | inflationary | <inflationadj> inflationary
    | exactly-soluble | unified | minimal | quantum | linear | nonlinear
    | gravitational | quantum gravitational | cosmological | supersymmetric
    | holographic | entropic | alternative | nonstandard | multidimensional
    | nonlocal | chiral | phenomenological | nonperturbative | perturbative
    | warped | <n>-dimensional | conformal | modified | supergravity mediated
    | gauge mediated | anomaly mediated | superconformal | extra-ordinary
    | general | anthropic | nilpotent | asymmetric | <symmetry> symmetric
    | <symmetry> invariant | spontaneous | thermodynamic | planar | inertial
    | metastable | unstable | tachyonic | transverse | longitudinal
    | momentum-dependent | exclusive | diffractive | dynamical | effective
    | acoustic | primordial | possible | impossible | calculable | predictive
    | unconventional | macroscopic | microscopic | holomorphic
    | consistent | inconsistent | anomalous
inflationadj ::= <inflationadj> <inflationadj> | <inflationadj> <inflationadj>
                | <inflationadj> <inflationadj> | <inflationadj> <inflationadj>
                | D-term | anisotropic | asymptotic | brane | braneworld chaotic
                | Brans-Dicke | chaotic | cosmological | de Sitter | double
                | dynamical | elastic | extended | extranatural | F-term | hybrid | false vacuum
                | first-order | general | generalized assisted | higher-curvature | hyper
                | inflatonless | inspired | inverted | K | large-scale | late-time
                | mild | low scale | modular invariant | multi-component | multi-field stochastic
                | multi-field | mutated | natural | new |  $\Omega_{<1>}$  | assisted | brane-assisted
                | tachyonic | liouville | open | Cobe-Dmr-normalized | D-term | dissipative

```

| supersymmetric | eternal | extended | extreme | facilitated | warm
 | generalized | gravitoelectromagnetic | holographic | induced | inhomogeneous
 | intermediate | kinetic | local | mass | moduli | slow-roll | multi-scalar
 | supergravity | natural | boundary | cosmic | dominated | early
 | exact | fake | field line | fresh | gravity driven | induced-gravity
 | intermediate scale | Jordan-Brans-Dicke | large field | locked
 | massive | monopole | multiple | multiple-stage | supergravity
 | non-slow-roll | old | particle physics | pole-like | power-law mass
 | precise | pseudonatural | quasi-open | racetrack | running-mass
 | simple | single scalar | single-bubble | spacetime | noncommutative
 | standard | steady-state | successful | sunergistic | tensor field
 | thermal brane | tilted ghost | topological | tsunami | unified | weak scale
 | noise-induced | one-bubble | open-universe | patch | polynomial | primary
 | quadratic | quintessential | rapid | asymmetric | scalar-tensor
 | non-canonical | smooth | spin-driven | Starobinsky | stochastic
 | string-forming | TeV-scale | three form | topological defect | viable
 | weak-dissipative | nonminimal | oscillating | phantom | power law
 | pre-big-bang | primordial | quantum | R-invariant | running
 | shear-free | rotating | slinky | spinodal | thermal | tidal | tree-level
 | two-stage | anthropic

Physicist

physicist ::= <physicistname> | <physicistname> | <physicistname>-<physicistname>
 physicistname ::=
 Weinberg | Feynman | Witten | Seiberg | Polchinski | Intrilligator
 | Vafa | Randall | Sundrum | Strominger | Georgi | Glashow | Coleman
 | Bohr | Fermi | Heisenberg | Maldacena | Einstein | Kachru | Arkani-Hamed
 | Schwinger | Higgs | Hitchin | Hawking | Stueckelberg | Unruh | Aranov-Bohm
 | 't Hooft | Silverstein | Horava | Lifschitz | Beckenstein | Planck
 | Euler | Lagrange | Maxwell | Boltzmann | Lorentz | Poincare | Susskind
 | Polyakov | Gell-Mann | Penrose | Dyson | Dirac | Argyres | Douglass
 | Gross | Politzer | Cabibo | Kobayashi | Denef | Shenker | Moore
 | Nekrosov | Gaiotto | Motl | Strassler | Klebanov | Nelson | Gubser
 | Verlinde | Bogoliubov | Schwartz

Concepts

mathconcept ::= <singmathconcept> | <pluralmathconcept>
 singmathconcept ::= integrability | perturbation theory | localization
 | duality | chaos | <mathadj> structure | <physicist>'s equation
 | dimensionality | <dualtype>-duality | unitarity
 | representation theory | Clebsch-Gordon decomposition
 | sheaf cohomology | anomaly matching |
 pluralmathconcept ::= gerbs | path integrals | Feynman diagrams | <mathadj> structures
 | <physicist> equations | <physicistname>'s equations | conformal blocks
 | <optype> operators | <dualtype>-dualities | <physicist> points
 | central charges | charges | currents | representations
 | <physicist> conditions | vortices | line bundles
 | symplectic quotients | hyperkahler quotients | Nahm's equations


```

    | vortex equations | Hilbert schemes | integration cycles | divisors
    | index theorems | flow equations | metrics | Gromov-Witten invariants
    | Gopakumar-Vafa invariants | Donaldson polynomials | <group> characters
physconcept ::= <pluralphysconcept> | <singphysconcept>
pluralphysconcept ::=
    examples of <physconcept> | equations of <theory>
    | <n>-point correlators | anomaly constraints | partition functions
    | <symmetry> algebras | fragmentation functions | decay constants
    | anomalous dimensions | PDFs | observables | effects of <physconcept>
    | <particle> collisions | <physadj> effects | <physadj> parameters
    | <physadj> hierarchies | <physconceptnoun> | <physadj> <physconceptnoun>
    | amplitudes | scattering amplitudes | geometric transitions
singphysconcept ::= <symmviol> <symmetry> invariance | <symmviol> <symmetry> symmetry
    | <symmetry> symmetry breaking | <mechanism> | confinement
    | the <physadj> limit | the partition function | the <physadj> formalism
    | the <physadj> law | the <symmetry> algebra | the beta function
    | the Wilsonian effective action | the <n>PI effective action
    | the partition function | <particle> production
    | the effective potential | the <particle> gyromagnetic ratio
    | renormalization | regularization | backreaction | AdS/CFT
    | a <physadj> hierarchy | the <physicist> formalism
    | <physadj> regularization | the 't Hooft anomaly matching condition
    | the S-matrix | the Hamiltonian | the Lagrangian
    | the omega deformation | the <physadj> Hilbert space
    | the Hilbert space | "<singphysconcept>" | <effect>
    | the OPE | IR behavior | UV behavior | a warped throat
    | a holographic superconductor | the <particle> charge
physconceptnoun ::= sectors | vacua | solutions | states | geometries | currents
    | backgrounds | wavefunctions | excitations | branching ratios
    | decays | exotics | corrections | interactions | inhomogeneities
    | correlation functions | amplitudes
dualtype ::= T|U|S|magnetic|electric|gravitational|boundary|Seiberg|Geometric Langlands
symmviol ::= | violation of | <physadj> violation of | breaking of
symmetry ::= dilation | translation | rotation | Lorentz | conformal | superconformal
    | super | Poincare | worldsheet | diffeomorphism | superdiffeomorphism
    | dual-superconformal | Yangian | Viroso | <liegroup>
mechanism ::= the <mechanismadj> mechanism | the <physadj> <mechanismadj> mechanism
mechanismadj ::= Higgs | seesaw | <physicist> | attractor | anomaly inflow | reheating
    | SuperHiggs | confinement
effect ::= the <effectadj> effect | the <physadj> <effectadj> effect | <physadj> effects
effectadj ::= <physicist> | quantum Hall | Unruh | Stark | Casimir

##### Subject #####

physsubject ::= <singphyssubject> | <pluralphyssubject>
singphyssubject ::=
    quintessence | <inflationadj> inflation | inflation | dark matter
    | spacetime foam | instanton gas | entropy | entanglement entropy | flavor
    | bubble nucleation | dark energy
pluralphyssubject ::=

```

```

condensates | <branetype> branes | cosmic rays | instanton liquids
| <physadj> fluctuations | bubbles
particle ::= hadron| lepton| quark| neutrino| electron| positron| WIMP|
slepton| squark| kk graviton| gluon| W-boson| Z-boson| neutralino|
chargino| ghost| axion| monopole| soliton| dion| kaon| B-meson| pion|
heavy-ion| Higgs
subject ::= <singsubject> | <pluralsubject>
pluralsubject ::= <pluralmodel> | <pluraltheoryobj> | <particle>s
| <pluralphysconcept> in <modeltheory>
| <pluralmathconcept> in <theory> | <mathadj> <pluralmathconcept>
| <pluralphysconcept> | <pluraltheory> | <pluralphyssubject> <objectplace>
| <pluraltheoryobj> <objectplace> | some <specific> <examples>
| <pluralmathconcept> on <space>
specific ::= specific | general | particular | conspicuous | little-known
examples ::= cases | examples | illustrations | computations | frameworks | paradigms
singsubject ::= <singmodel> | <singtheory> | <singtheoryobj> | <problem>
| <solution> | <studyingverb> <modeltheory>
| <article> <physadj> <actiondone> of <modeltheory>
| <singphysconcept> in <modeltheory> | <dualtype>-duality in <modeltheory>
| <singmathconcept> in <theory> | <mathadj> <singmathconcept>
| <singphysconcept> | the <actiondone> of <modeltheory>
| <article> <actiondone> of <mathconcept> in <modeltheory>
| the <correspondent>/<correspondent> correspondence
| <article> <dualtype>-dual of <modeltheory>
| <singtheoryobj> <objectplace> | <singphyssubject> <objectplace>
| <singsubject> (<including> <subject>) | <singmathconcept> on <space>
| <singmathconcept> | a certain notion of <singmathconcept>
modeltheory ::= <model> | <theory>
including ::= including | excluding | involving | taking into account
correspondent ::= <generalmodel> | <propermodel> | <properqft> | <genericqft> | <mathspace>
solution ::= <article> solution <solved> | <article> <soladj> solution <solved>
| <article> solution <solved> <via> <subject>
| <article> <soladj> solution <solved> <via> <subject>
| a resolution of <problem> | a <soladj> resolution of <problem>
| a <soladj> approach to <problem>
solved ::= to <problem> | of <theory>
via ::= via | through | from | by
soladj ::= better | new | beautiful | quantum | physical | old | clever
| minimal | non-minimal | <physadj> | anthropic | entropic | possible
| probable | partial
problem ::= the <problemtype> problem
problemtype ::= hierarchy | flavor | cosmological constant | lithium | mu
| strong CP | naturalness | little hierarchy | SUSY CP | LHC inverse
| cosmic coincidence | U(1) | fine-tuning | mu/B_mu | confinement

##### Verbs #####

verb ::= derive | obtain | deduce | discover | find
| conjecture | check | calculate | predict
verbed ::= derived | obtained | deduced | discovered | found | conjectured

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    | realized | checked | calculated | predicted
studyverb ::= study | solve | investigate | demystify | bound
    | classify | obtain | derive | generalize | explore
    | examine | consider | analyze | evaluate | review
    | survey | explain | clarify | shed light on
    | extend | construct | reconstruct | calculate | discuss
    | formulate | reformulate | understand
studyingverb ::= studying | solving | investigating | demystifying | bounding
    | classifying | obtaining | deriving | generalizing | exploring
    | examining | considering | analyzing | evaluating | reviewing
    | surveying | explaining | clarifying | formulating | reformulating
    | extending | constructing | reconstructing | discussing | understanding
studiedverb ::= studied | solved | investigated | demystified | bounded
    | classified | obtained | derived | generalized | explored
    | examined | considered | analyzed | evaluated | reviewed
    | surveyed | recalled | explained | clarified | extended | constructed
    | reconstructed | discussed | understood
singbeingverb ::= exists | is present | must be there | must be present | does not exist
revealed ::= revealed | produced | led to | led us to | exposed | uncovered
singstatementverb ::= is | is equivalent to | is related to | derives from
    | reduces to | follows from | lets us <studyverb>
    | can be interpreted as | can be <verbed> from | turns out to be equivalent to
    | relates to | depends on | <adverb> <singstatementverb>
    | can be incorporated into | can be brought to bear in <studyingverb>
    | is useful for <studyingverb> | is the final component in <studyingverb>
pluralstatementverb ::= are the same as | are equivalent to | are related to
    | let us <studyverb> | can compute | follow from | can be interpreted as
    | can be <verbed> from | turn out to be equivalent to | relate to | depend on
    | derive from | reduce to | <adverb> <pluralstatementverb>
    | can be incorporated into | can be brought to bear in <studyingverb>
    | are useful for <studyingverb> | relate <subject> to
yields ::= yields | gives | provides | produces | gives rise to
prove ::= prove | show | demonstrate | establish | illustrate
    | determine | confirm | verify
contradict ::= contradict | disagree with | agree with
    | find inconsistencies with | argue against
    | run counter to | cannot corroborate | cannot support
    | challenge | fail to <prove>

```

Language

```

capital ::= A|B|C|D|E|F|G|H|I|J|K|L|M|N|O|P|Q|R|S|T|U|V|W|X|Y|Z
article ::= a | the
adverb ::= remarkably | actually | interestingly | however | moreover
    | therefore | thus | surprisingly | unsurprisingly
    | consequently | curiously | fortunately | unfortunately
    | quite simply | in short
recently ::= recently | in recent years | in recent papers | over the last decade
    | in the 20th century | among particle physicists | among mathematicians
thereby ::= <thereby> <thereby> | thereby | completely | conclusively | wholly

```

| thoroughly | fully | ultimately | unambiguously
 motivated ::= motivated by this | inspired by this | continuing in this vein
 | continuing with this program
 assuming ::= if | whenever | provided that | supposing that | assuming | assuming that
 | as long as | given that
 preposition ::= after | before | while | when
 whenphrase ::= <preposition> <studyingverb> <subject>
 actiondone ::= reduction | compactification | formulation
 | extension | solution | analytic continuation
 qualifier ::= at least in the context of <subject> | without regard to <subject>
 | in the approximation that <statement> | in the limit that <statement>
 | as realized in <subject> | as hinted at by <physicist>
 | as revealed by <mathconcept> | by <symmetry> symmetry | by symmetry
 | whenever <statement> | as we will see in this paper
 | with the help of <subject> | in the <singmathconcept> case
 | as will be made clear | as will be <studiedverb> shortly
 inorderto ::= to <prove> that <statement> | in order to <prove> that <statement>
 | in order to avoid <studyingverb> <subject>
 | to best <studyverb> <subject>
 | to <studyverb> <subject> | in a way that <yields> <subject>
 | to <studyverb> recent results linking <subject> and <subject>
 | to explore questions such as the <singmathconcept> conjecture
 was ::= has been | was
 muchwork ::= much work <was> done | interesting progress <was> made
 | substantial progress has been made | minimal progress <was> made
 | some work <was> done | little work <was> done
 | a fair amount of work <was> done
 | partial progress <was> made
 test ::= <computation> | test | probe | measurement | check
 computation ::= computation | calculation | determination
 correspondence ::= correspondence | conjecture | theorem | result
 fact ::= fact | truth | principle | law | theorem | rule | pattern
 | structure | framework | edifice
 thesame ::= the same | the very same | our very same
 | our | the exact same | a previously studied
 beautiful ::= beautiful | surprising | elegant | pretty | arresting | charming
 | simple | ingenious | sophisticated | intricate | elaborate | detailed
 | confusing | bewildering | perplexing | elaborate | involved | complicated
 | startling | unforeseen | amazing | extraordinary | remarkable
 | shocking | unexpected | deep | mysterious | profound | unsurprising
 | essential | fundamental | crucial | critical | key | important

Statements & Sentences

statement ::= <singsubject> <singstatementverb> <singsubject>
 | <pluralsubject> <pluralstatementverb> <subject>
 | <singsubject> is <descriptivephysadj>
 | <pluralsubject> are <descriptivephysadj>
 asentence ::=
 <asentence>, <qualifier>

```

| <recently>, <muchwork> on <model>
| <recently>, <muchwork> <studyingverb> <theory>
| <recently>, <muchwork> on <model> <inorderto>
| <recently>, <muchwork> <studyingverb> <theory> <inorderto>
| <muchwork> <recently> on <model>
| <muchwork> <recently> <studyingverb> <theory>
| <recently>, work on <model> has opened up a <descriptivephysadj>
  class of <physadj> models
| <recently>, <physicistname> <studiedverb> <subject>
| <recently>, <physicistname> <verbed> that <statement>
| <asentence>. we take a <descriptivephysadj> approach
| <asentence>. <motivated>, <bsentence>
| <singsubject> offers the possibility of <studyingverb> <subject>
| <subject> <yields> a <beautiful> framework for <studyingverb> <subject>
| <singsubject> is usually <verbed> <via> <subject>
| <pluralsubject> are usually <verbed> <via> <subject>
bsentence ::=
  <bsentence>, <qualifier>
| <inorderto>, <bsentence>
| we <studyverb> <subject>
| we solve <problem>
| we take a <descriptivephysadj> approach to <subject>
| we <prove> that <statement>
| we <prove> a <beautiful> correspondence between <subject> and <subject>
| <bsentence>, and <studyverb> <subject>
| <bsentence>, and <verb> that <statement>
| <bsentence>, and <verb> that, <qualifier>, <statement>
| <bsentence>, <thereby> <studyingverb> that <statement>
| <via> <studyingverb> <pluralmathconcept>, we <studyverb> <subject>
| <via> <studyingverb> <physconcept>, we <studyverb> <subject>
| we <verb> evidence for <subject>
| using the behavior of <singsubject>, we <studyverb> <subject>
| we present a criterion for <subject>
| we make contact with <subject>, <adverb> <studyingverb> <subject>
| we make contact between <subject> and <subject>
| we <studyverb> why <statement>
| we use <subject> to <studyverb> <subject>
| we use <subject>, together with <subject> to <studyverb> <subject>
| in this paper, <bsentence>
csentence ::=
  <csentence>, <qualifier>
| <motivated>, <bsentence>
| we take a <descriptivephysadj> approach
| <adverb>, <statement>
| next, <bsentence>
| <singtheory> is also <studiedverb>
| <pluraltheory> are also <studiedverb>
| <singmodel> is also <studiedverb>
| <pluralmodel> are also <studiedverb>
| <singphysconcept> is also <studiedverb>

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| <pluralphysconcept> are also <studiedverb>
| we <thereby> <prove> a <beautiful> correspondence between <subject> and <subject>
| we also <verb> agreement with <subject>
| the <computation> of <physconcept> localizes to <space>
| <statement> <assuming> <statement>
| <subject> <revealed> a <beautiful> <fact>: <statement>
| <studyingverb> is made easier by <studyingverb> <subject>
| our <computation> of <subject> <yields> <subject>
| as an interesting outcome of this work for <subject>, <bsentence>
| <csentence>, <studyingverb> <subject>
| <adverb>, <singsubject> <singstatementverb> <thesame> <singmathconcept>
| we therefore <contradict> a result of <physicistname> that <statement>
| this probably <singstatementverb> <subject>, though we've been unable
| to <prove> a <correspondence>
| this is most likely a result of <physsubject>, an observation first
| mentioned in work on <subject>
| this <yields> an extremely precise <test> of <singphysconcept>
| the <singmathconcept> depends, <adverb>, on whether <statement>
| a <beautiful> part of this analysis <singstatementverb> <subject>
| in this <correspondence>, <singsubject> makes a <beautiful> appearance
| why this happens can be <studiedverb> by <studyingverb> <subject>
| the title of this article refers to <subject>
| we <verb> that <singtheoryobj> <singbeingverb> <qualifier>
| this <correspondence> has long been understood in terms of <subject>
dsentence ::=
  <dsentence>, <qualifier>
  | <whenphrase>, we <verb> that <statement> | <statement>
  | <whenphrase>, we <verb> that, <qualifier>, <statement>
  | <dsentence>. <adverb>, <dsentence> | our results <prove> that <statement>
closing ::= finally, <bsentence>
  | <adverb>, there is much to be done
  | we hope this paper provides a good starting point for <studyingverb> <subject>
  | we leave the rest for future study
  | <adverb>, <singsubject> is beyond the scope of this paper
  | we will provide more details in a future paper
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